

ADELIC CARTIER DIVISORS WITH BASE CONDITIONS AND THE CONTINUITY OF VOLUMES

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ABSTRACT. In the previous paper [7], we introduced a notion of pairs of adelic \mathbb{R} -Cartier divisors and \mathbb{R} -base conditions. The purpose of this paper is to propose an extended notion of adelic \mathbb{R} -Cartier divisors that we call an ℓ^1 -adelic \mathbb{R} -Cartier divisors, and to show that the arithmetic volume function defined on the space of pairs of ℓ^1 -adelic \mathbb{R} -Cartier divisors and \mathbb{R} -base conditions is continuous along the directions of ℓ^1 -adelic \mathbb{R} -Cartier divisors.

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1. INTRODUCTION

In Arakelov geometry, it is essentially important whether or not an adelic line bundle has a nonzero small section. The asymptotic number of the small sections of high powers of an adelic line bundle \overline{L} is encoded in an invariant which we call the *arithmetic volume* of \overline{L} and denote by $\widehat{\text{vol}}(\overline{L})$. The notion of arithmetic volume was first introduced by Moriwaki in a series of papers [11, 12, 14], where he proved that the arithmetic volume has many good properties such as the global continuity, the positive homogeneity, the birational invariance, etc. A purpose of this paper is to give a generalization of Moriwaki's arithmetic volume function, and study its fundamental properties.

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Let K be a number field, and let O_K be the ring of integers of K . Let M_K^{fin} be the set of all the finite places of K . For each $v \in M_K^{\text{fin}}$, K_v denotes the v -adic completion of K , and \tilde{K}_v denotes the residue field at v . Let X be a normal projective K -variety, and let $\text{Rat}(X)$ be the field of rational functions on X . For each $v \in M_K^{\text{fin}} \cup \{\infty\}$, let X_v^{an} be the associated analytic space over v (see section 3.1.2 for detail). Let D be an \mathbb{R} -Cartier divisor on X endowed with a D -Green function g_∞ on X_∞^{an} . To an O_K -model $(\mathcal{X}, \mathcal{D})$ of (X, D) , we can associate an adelic \mathbb{R} -Cartier divisor

$$(\mathcal{D}, g_\infty)^{\text{ad}} := \left(D, \sum_{v \in M_K^{\text{fin}}} g_v^{(\mathcal{X}, \mathcal{D})}[v] + g_\infty[\infty] \right).$$

We then define the ℓ^1 -distance of two such models $(\mathcal{X}_1, \mathcal{D}_1)$ and $(\mathcal{X}_2, \mathcal{D}_2)$ as

$$\sum_{v \in M_K^{\text{fin}}} \sup_{x \in X_v^{\text{an}}} \left| g_v^{(\mathcal{X}_1, \mathcal{D}_1)}(x) - g_v^{(\mathcal{X}_2, \mathcal{D}_2)}(x) \right|.$$

For example, let v_1, v_2, \dots be a sequence in M_K^{fin} , and let \mathcal{F}_i be the fiber of \mathcal{X} over v_i . The sequence of O_K -models

$$\left(\left(\mathcal{X}, \sum_{i=1}^n \frac{1}{2^i \log \# \tilde{K}_{v_i}} \mathcal{F}_i \right) \right)_{n \geq 1}$$

is then a Cauchy sequence in the ℓ^1 -distance. However, it does not have a limit in the space of adelic \mathbb{R} -Cartier divisors. A basic principle of functional analysis tells us that function spaces should be complete, so we decide to extend the notion of adelic \mathbb{R} -Cartier divisors so as the above sequence is to converge. For each $v \in M_K^{\text{fin}} \cup \{\infty\}$, we put $C(X_v^{\text{an}})$ as the Banach algebra of \mathbb{R} -valued continuous functions on X_v^{an} endowed with the supremum norm. If $v = \infty$, we impose the condition that the functions in $C(X_\infty^{\text{an}})$ are invariant under the complex conjugation map. We define the space $C_{\ell^1}(X)$ of continuous functions on X as the ℓ^1 -direct sum of the family $(C(X_v^{\text{an}}))_{v \in M_K^{\text{fin}} \cup \{\infty\}}$ endowed with the ℓ^1 -norm $\|\cdot\|_{\ell^1}$. We say that a couple $\overline{D} = \left(D, \sum_{v \in M_K^{\text{fin}} \cup \{\infty\}} g_v[v] \right)$ of an \mathbb{R} -Cartier divisor D and an adelic D -Green function $\sum_{v \in M_K^{\text{fin}} \cup \{\infty\}} g_v[v]$ is an ℓ^1 -adelic \mathbb{R} -Cartier divisor if there exists an O_K -model $(\mathcal{X}, \mathcal{D})$ of (X, D) such that $\|\overline{D} - (\mathcal{D}, g_\infty)^{\text{ad}}\|_{\ell^1} < +\infty$, and denote by $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ the \mathbb{R} -vector space of all the ℓ^1 -adelic \mathbb{R} -Cartier divisors on X .

There are several advantages of such an extension. For example, the quotient space $\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ of $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ by the \mathbb{R} -subspace generated by principal adelic Cartier divisors admits an essentially unique norm that makes $\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ into a Banach space (see section 3.3), which should be a proper arithmetic analogue of the space of numerical classes of \mathbb{R} -Cartier divisors in algebraic geometry. In particular, any surjective natural homomorphism $\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X) \rightarrow \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(Y)$ is automatically an open mapping. We expect that such a formalism will open a way for applying the powerful machinery of functional analysis, such as the duality theory, the semigroup theory, the spectral theory, etc., to the study of adelic \mathbb{R} -Cartier divisors.

In the previous paper [7], we introduced a notion of \mathbb{R} -base conditions, and defined the arithmetic volumes for pairs of adelic \mathbb{R} -Cartier divisors and \mathbb{R} -base

conditions. An \mathbb{R} -base condition \mathcal{V} on X is defined as a formal \mathbb{R} -linear combination

$$\mathcal{V} = \sum_{\nu} \nu(\mathcal{V})[\nu]$$

such that ν are normalized discrete valuations of $\text{Rat}(X)$ and such that $\nu(\mathcal{V})$ are zero for all but finitely many ν . A discrete valuation ν assigns to D an *order of vanishing along ν* defined as $\nu(f)$, where f is a local equation defining D around the center $c_X(\nu)$ of ν on X . We denote by $\text{BC}_{\mathbb{R}}(X)$ the \mathbb{R} -vector space of all the \mathbb{R} -base conditions on X . Given a pair $(\overline{D}; \mathcal{V})$ of $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$, we can define

$$\widehat{\ell}^s(\overline{D}; \mathcal{V}) := \log \left(1 + \sharp \left\{ \phi \in \text{Rat}(X)^{\times} : \overline{D} + (\widehat{\phi}) > 0, \nu(D + (\phi)) \geq \nu(\mathcal{V}), \forall \nu \right\} \right)$$

as a nonnegative real number (see Proposition 3.12), and can define the *arithmetic volume* of $(\overline{D}; \mathcal{V})$ as

$$\widehat{\text{vol}}(\overline{D}; \mathcal{V}) := \lim_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!}$$

(see Proposition 3.13). We will establish the following result (Theorem 3.21).

Main Theorem. *Let X be a normal, projective, and geometrically connected K -variety. Let V be a finite-dimensional \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$, let $\|\cdot\|_V$ be a norm on V , let Σ be a finite set of points on X , and let $B \in \mathbb{R}_{>0}$. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\left| \widehat{\text{vol}}(\overline{D} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\text{vol}}(\overline{E}; \mathcal{V}) \right| \leq \varepsilon$$

for every $\overline{D}, \overline{E} \in V$ with $\max \{ \|\overline{D}\|_V, \|\overline{E}\|_V \} \leq B$ and $\|\overline{D} - \overline{E}\|_V \leq \delta$, $\mathbf{f} \in C_{\ell^1}(X)$ with $\|\mathbf{f}\|_{\ell^1} \leq \delta$, and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ with $\{c_X(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma$.

This paper comprises two parts. First, in section 2, after showing preliminary results on base conditions (section 2.1) and the change of norms (section 2.2), we prove in section 2.3 the fundamental estimate of numbers of small sections of pairs, which is the key step to show Theorem 3.21.

Next, section 3 will be devoted to introducing the notion of ℓ^1 -adelic \mathbb{R} -Cartier divisors and showing Theorem 3.21. After recalling basic facts on the adelicly normed vector spaces (section 3.1.1), the Berkovich analytic spaces (section 3.1.2), and the D -Green functions (section 3.1.3), we will introduce basic definitions on the ℓ^1 -adelic setting in sections 3.2 and 3.3. We will define the arithmetic volumes of pairs of ℓ^1 -adelic \mathbb{R} -Cartier divisors and \mathbb{R} -base conditions in section 3.4 and give a proof of Theorem 3.21 in section 3.5.

1.1. Notation and terminology.

1.1.1. Let R be a ring and let M be an R -module. Given a subset Γ of M , we denote by $\langle \Gamma \rangle_R$ the R -submodule of M spanned by Γ . In this paper, we adopt the dot-product notation, that is, for $\mathbf{a} = (a_1, \dots, a_r) \in R^r$ and $\mathbf{m} = (m_1, \dots, m_r) \in M^r$, we write

$$\mathbf{a} \cdot \mathbf{m} = a_1 m_1 + \dots + a_r m_r.$$

Moreover, for $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}^r$, $\|\mathbf{a}\|_1$ denotes the ℓ^1 -norm of \mathbf{a} :

$$\|\mathbf{a}\|_1 := |a_1| + \dots + |a_r|.$$

1.1.2. A *normed \mathbb{Z} -module* $\overline{M} := (M, \|\cdot\|)$ is a finitely generated \mathbb{Z} -module M endowed with a norm $\|\cdot\|$ on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For such an \overline{M} , we set

$$\widehat{\Gamma}^s(\overline{M}) := \{m \in M : \|m \otimes 1\| \leq 1\}, \quad \widehat{\ell}^s(\overline{M}) := \log \# \widehat{\Gamma}^s(\overline{M})$$

and

$$\widehat{\Gamma}^{ss}(\overline{M}) := \{m \in M : \|m \otimes 1\| < 1\}, \quad \widehat{\ell}^{ss}(\overline{M}) := \log \# \widehat{\Gamma}^{ss}(\overline{M}).$$

Let $*$ = s or ss. The following properties are fundamental.

(a) Let \overline{M} be a normed \mathbb{Z} -module and let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be an exact sequence of \mathbb{Z} -modules. We endow $M'_{\mathbb{R}}$ (respectively, $M''_{\mathbb{R}}$) with the subspace norm $\|\cdot\|_{\text{sub}}$ (respectively, quotient norm $\|\cdot\|_{\text{quot}}$) induced from \overline{M} . One then has

$$(1.1) \quad \widehat{\ell}^*(\overline{M}) \leq \widehat{\ell}^*(\overline{M}') + \widehat{\ell}^*(\overline{M}'') + 3 \operatorname{rk} M' + 2 \log(\operatorname{rk} M')!$$

In fact, if $*$ = s, then the inequality is nothing but [11, Proposition 2.1(4)] and, if $*$ = ss, then it follows from the $*$ = s case by replacing $\|\cdot\|$ with $e^{\varepsilon} \|\cdot\|$ for $\varepsilon > 0$ and taking $\varepsilon \downarrow 0$.

(b) If we replace $\|\cdot\|$ with $e^{-\lambda} \|\cdot\|$ for a $\lambda \in \mathbb{R}_{\geq 0}$, then

$$(1.2) \quad \widehat{\ell}^*(M, \|\cdot\|) \leq \widehat{\ell}^*(M, e^{-\lambda} \|\cdot\|) \leq \widehat{\ell}^*(M, \|\cdot\|) + (\lambda + 2) \operatorname{rk} M$$

(see the proof of [16, Lemma 2.9]).

(c) If M' is a \mathbb{Z} -submodule of M with M/M' torsion, then

$$(1.3) \quad \widehat{\ell}^*(M', \|\cdot\|) \leq \widehat{\ell}^*(M, \|\cdot\|) \leq \widehat{\ell}^*(M', \|\cdot\|) + \log \#(M/M') + 2 \operatorname{rk} M$$

(see [14, Lemma 1.3.3, (1.3.3.4)]).

(d) Let M be a finitely generated \mathbb{Z} -module, and let $\|\cdot\|^1, \|\cdot\|^2$ be two norms on $M_{\mathbb{R}}$. If $\|\cdot\|^1 \leq \|\cdot\|^2$, then

$$(1.4) \quad \widehat{\ell}^*(M, \|\cdot\|^1) \geq \widehat{\ell}^*(M, \|\cdot\|^2).$$

(e) Let $\langle \cdot, \cdot \rangle^1$ and $\langle \cdot, \cdot \rangle^2$ be two Hermitian inner products on $M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C}$, and let $\|\cdot\|^1$ and $\|\cdot\|^2$ be the associated norms on $M_{\mathbb{R}}$, respectively. Let e_1, \dots, e_l be any basis for $M_{\mathbb{C}}$. If $\|\cdot\|^1 \leq \|\cdot\|^2$, then

$$(1.5) \quad \widehat{\ell}^*(M, \|\cdot\|^1) - \widehat{\ell}^*(M, \|\cdot\|^2) - 3 \operatorname{rk} M - 2 \log(\operatorname{rk} M)! \\ \leq -\frac{1}{2} \log \frac{\det(\langle e_i, e_j \rangle^1)_{i,j}}{\det(\langle e_i, e_j \rangle^2)_{i,j}}$$

(see [11, Proposition 2.1(2)]). The right-hand side does not depend on a specific choice of e_1, \dots, e_l . The $*$ = ss case follows by the same arguments as in (a) above.

We will also use the elementary inequalities

$$\log n! \leq n \log n \quad \text{and} \quad \log n \leq n$$

for every $n \in \mathbb{Z}_{>0}$.

1.1.3. Let k be a field endowed with a non-Archimedean absolute value $|\cdot|$. We write

$$(1.6) \quad k^\circ := \{a \in k : |a| \leq 1\}, \quad k^{\circ\circ} := \{a \in k : |a| < 1\}, \quad \text{and} \quad \tilde{k} := k^\circ/k^{\circ\circ}.$$

1.1.4. Let K be a number field and let O_K be the ring of integers of K . Let M_K^{fin} be the set of all the finite places of K and set

$$(1.7) \quad M_K := M_K^{\text{fin}} \cup \{\infty\}.$$

Set $K_\infty := \mathbb{C}$ and set $|\alpha|_\infty := \sqrt{\alpha\bar{\alpha}}$ for $\alpha \in \mathbb{C}$. For $v \in M_K^{\text{fin}}$, we denote by \mathfrak{p}_v the prime ideal of O_K corresponding to v , by $K_v^\circ = \text{proj} \lim_{n \in \mathbb{Z}_{>0}} O_K/\mathfrak{p}_v^n$ the v -adic completion of O_K , and by K_v the quotient field of K_v° . We put

$$(1.8) \quad K_v^{\circ\circ} := \mathfrak{p}_v K_v^\circ \quad \text{and} \quad \tilde{K}_v := K_v^\circ/K_v^{\circ\circ}.$$

We will write a uniformizer of K_v by ϖ_v . We define the order of an $\alpha \in K_v^\circ$ as

$$(1.9) \quad \text{ord}_v(\alpha) := \begin{cases} \max \{n \geq 0 : \alpha \in (K_v^{\circ\circ})^n\} & \text{if } \alpha \neq 0 \text{ and} \\ +\infty & \text{if } \alpha = 0, \end{cases}$$

and extend it to a map from K_v by linearity. The (normalized) v -adic absolute value on K_v is defined as

$$(1.10) \quad |\alpha|_v := \left(\#\tilde{K}_v\right)^{-\text{ord}_v(\alpha)}$$

for $\alpha \in K_v$.

2. FUNDAMENTAL ESTIMATE

2.1. Base conditions.

2.1.1. Let F be a field. A *normalized discrete valuation* ν on F is a surjective map from F to $\mathbb{Z} \cup \{+\infty\}$ such that

- (a) $\nu(f) = +\infty$ if and only if $f = 0$,
- (b) $\nu(f \cdot g) = \nu(f) + \nu(g)$ for $f, g \in F$, and
- (c) $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for $f, g \in F$.

We set $F_\nu^\circ := \{f \in F : \nu(f) \geq 0\}$ and $F_\nu^{\circ\circ} := \{f \in F : \nu(f) > 0\}$. Since $(F_\nu^\circ)^\times = \{f \in F : \nu(f) = 0\}$, $F_\nu^{\circ\circ}$ is a maximal ideal of F_ν° . We denote by $\mathfrak{V}(F)$ the set of all the normalized discrete valuations on F .

2.1.2. Let S be a reduced, irreducible, and separated scheme and let $F := \text{Rat}(S)$ be the field of rational functions on S . We assume the condition that,

- (\star) for every $\nu \in \mathfrak{V}(F)$, there exists a unique point $c_S(\nu) \in S$ such that

$$\mathcal{O}_{S, c_S(\nu)} \subset F_\nu^\circ \quad \text{and} \quad \mathfrak{m}_{c_S(\nu)} = F_\nu^{\circ\circ} \cap \mathcal{O}_{S, c_S(\nu)}.$$

We call $c_S(\nu)$ the *center* of ν on S . By the valuative criterion of properness, if S is proper over $\text{Spec}(\mathbb{Z})$, then S satisfies the condition (\star).

Remark 2.1. If S is a proper variety over a field k , then we always assume that a valuation $\nu \in \mathfrak{V}(F)$ is trivial on k . In particular, such a valuation always has a unique center $c_S(\nu)$ on S , and the condition (\star) is satisfied.

An \mathbb{R} -base condition \mathcal{V} on S is defined as a finite formal sum

$$\mathcal{V} := \sum_{\nu \in \mathfrak{V}(\text{Rat}(S))} \nu(\mathcal{V})[\nu]$$

with real coefficients $\nu(\mathcal{V})$. We denote by $\text{BC}_{\mathbb{R}}(S)$ the \mathbb{R} -vector space of all the \mathbb{R} -base conditions on S . We write $\mathcal{V} \geq 0$ if $\nu(\mathcal{V}) \geq 0$ for every $\nu \in \mathfrak{V}(\text{Rat}(S))$.

2.1.3. Let S be a reduced, irreducible, and projective scheme over a field or \mathbb{Z} . Let L be a line bundle on S , let $\nu \in \mathfrak{V}(\text{Rat}(S))$, and let η be a local frame of L around $c_S(\nu)$. Given any $s \in H^0(L) \setminus \{0\}$, one can write $s_{c_S(\nu)} = f\eta_{c_S(\nu)}$ with $f \in \mathcal{O}_{S, c_S(\nu)} \setminus \{0\}$. If η' is another local frame of L around $c_S(\nu)$, then η'/η is invertible in $\mathcal{O}_{S, c_S(\nu)}$. So, if we write $s_{c_S(\nu)} = f'\eta'_{c_S(\nu)}$ with $f' \in \mathcal{O}_{S, c_S(\nu)} \setminus \{0\}$, then f/f' is invertible in $\mathcal{O}_{S, c_S(\nu)}$ and $\nu(f) = \nu(f')$. We define

$$(2.1) \quad \nu(s) := \nu(f),$$

which does not depend on a specific choice of η . The following properties are obvious.

- (a) If $s \in H^0(L)$ does not pass through $c_S(\nu)$, then f is invertible around $c_S(\nu)$ and

$$(2.2) \quad \nu(s) = 0.$$

- (b) For $s, t \in H^0(L)$ and $\nu \in \mathfrak{V}(\text{Rat}(S))$,

$$(2.3) \quad \nu(s + t) \geq \min\{\nu(s), \nu(t)\}.$$

- (c) For two line bundles L, M on S , $s \in H^0(L)$, $t \in H^0(M)$, and $\nu \in \mathfrak{V}(\text{Rat}(S))$, one has

$$(2.4) \quad \nu(s \otimes t) = \nu(s) + \nu(t).$$

For a pair $(L; \mathcal{V})$ of a line bundle L and a $\mathcal{V} \in \text{BC}_{\mathbb{R}}(S)$, we set

$$(2.5) \quad H^0(L; \mathcal{V}) := \{s \in H^0(L) : \nu(s) \geq \nu(\mathcal{V}) \text{ for all } \nu \in \mathfrak{V}(\text{Rat}(S))\}.$$

2.1.4. Let k be a field or \mathbb{Z} . Let S be a reduced, irreducible, normal, and projective k -scheme, and let $\mathbb{K} = \mathbb{R}, \mathbb{Q}$, or \mathbb{Z} . A \mathbb{K} -Cartier divisor on S is an \mathbb{K} -linear combination

$$D = a_1 D_1 + \cdots + a_r D_r$$

such that $a_i \in \mathbb{K}$ and such that D_i are Cartier divisors. We denote by $\text{Div}_{\mathbb{K}}(S)$ the \mathbb{K} -module of all the \mathbb{K} -Cartier divisors on S . If $\mathbb{K} = \mathbb{Z}$, we simply write $\text{Div}(S) := \text{Div}_{\mathbb{Z}}(S)$ as usual.

Each $\nu \in \mathfrak{V}(\text{Rat}(S))$ can extend to a map $\nu : \text{Rat}(S)^{\times} \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}$ by linearity. Given a $D \in \text{Div}_{\mathbb{R}}(S)$ and a $\nu \in \mathfrak{V}(\text{Rat}(S))$, we take a local equation f defining D around $c_S(\nu)$, and define

$$(2.6) \quad \nu(D) := \nu(f),$$

which does not depend on a specific choice of f (see [7, Definition 2.2]). Given a pair $(D; \mathcal{V})$ of an \mathbb{R} -Cartier divisor D and a $\mathcal{V} \in \text{BC}_{\mathbb{R}}(S)$, we set

$$(2.7) \quad H^0(D; \mathcal{V}) := \left\{ \phi \in \text{Rat}(S)^{\times} : \begin{array}{l} D + (\phi) \geq 0 \text{ and } \nu(D + (\phi)) \geq \nu(\mathcal{V}) \\ \text{for all } \nu \in \mathfrak{V}(\text{Rat}(S)) \end{array} \right\} \cup \{0\},$$

and define

$$(2.8) \quad \text{vol}(D; \mathcal{V}) := \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\text{rk}_k H^0(mD; m\mathcal{V})}{m^{\dim S} / (\dim S)!}.$$

2.1.5. Let \mathcal{X} be a projective arithmetic variety over $\text{Spec}(\mathbb{Z})$; namely, \mathcal{X} is a reduced and irreducible scheme projective and flat over $\text{Spec}(\mathbb{Z})$. Let $\mathcal{X}(\mathbb{C})$ be the complex analytic space associated to $\mathcal{X}_{\mathbb{C}} := \mathcal{X} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{C})$. A continuous Hermitian line bundle on \mathcal{X} is a couple $(\mathcal{L}, |\cdot|_{\overline{\mathcal{L}}})$ of a line bundle \mathcal{L} on \mathcal{X} and a continuous Hermitian metric $|\cdot|_{\overline{\mathcal{L}}}$ on $\mathcal{L}(\mathbb{C})$.

Definition 2.1. Let $(\overline{\mathcal{L}}; \mathcal{V})$ be a pair of a continuous Hermitian line bundle $\overline{\mathcal{L}}$ on \mathcal{X} and a $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$. The \mathbb{Z} -module

$$H^0(\mathcal{L}; \mathcal{V}) = \{s \in H^0(\mathcal{L}) : \nu(s) \geq \nu(\mathcal{V}) \text{ for all } \nu \in \mathfrak{V}(\text{Rat}(\mathcal{X}))\}$$

is endowed with the *supremum norm* $\|\cdot\|_{\sup}^{\overline{\mathcal{L}}}$ defined as

$$(2.9) \quad \|s\|_{\sup}^{\overline{\mathcal{L}}} := \sup_{x \in \mathcal{X}(\mathbb{C})} |s|_{\overline{\mathcal{L}}}(x)$$

for $s \in H^0(\mathcal{L})$. We will abbreviate

$$(2.10) \quad \widehat{\ell}^*(\overline{\mathcal{L}}; \mathcal{V}) := \widehat{\ell}^*\left(H^0(\mathcal{L}; \mathcal{V}), \|\cdot\|_{\sup}^{\overline{\mathcal{L}}}\right)$$

for $*$ = s and ss (see Notation and terminology 1.1.2).

2.2. Comparison of norms. Let T be a finite disjoint union

$$T = \bigcup_{i=1}^l T_i$$

of compact complex Kähler manifolds T_i of pure dimension d . Let ω be a Kähler form on T and let $\Omega = \omega^{\wedge d}$ be the volume form on T associated to ω . Let $\overline{M} = (M, h^{\overline{M}})$ be a line bundle M on T endowed with a C^∞ -Hermitian metric $h^{\overline{M}}$. The *supremum norm* of $s \in H^0(M)$ is defined as

$$\|s\|_{\sup}^{\overline{M}} := \sup_{t \in T} |s|_{\overline{M}}(t), \quad \text{where} \quad |s|_{\overline{M}}(t) := \sqrt{h^{\overline{M}}(s, s)(t)}.$$

The L^2 -inner product of $s_1, s_2 \in H^0(M)$ with respect to Ω is defined as

$$\langle s_1, s_2 \rangle_{L^2}^{\overline{M}} := \int_T h^{\overline{M}}(s_1, s_2)(t) \Omega,$$

and the L^2 -norm of s is $\|s\|_{L^2}^{\overline{M}} := \sqrt{\langle s, s \rangle_{L^2}^{\overline{M}}}$. In the rest of this subsection, we study the effects of the change of norms to the numbers of small sections.

2.2.1. The first one (Proposition 2.3) gives us a direct (not optimal) relation between the supremum norms and the subspace norms induced by a fixed nonzero section.

Lemma 2.2 (see [11, Lemma 1.1.4]). *Let $\overline{\mathbf{M}} = (\overline{M}_1, \dots, \overline{M}_r)$ be C^∞ -Hermitian line bundles on T and let U be an open subset of T . Assume that $U \cap T_i$ are*

nonempty for all i . There then exists a positive constant $C_1 \geq 1$ depending only on \overline{M} , U , and T such that

$$\sup_{t \in U} |s|^{\mathbf{a} \cdot \overline{M}}(t) \leq \|s\|_{\sup}^{\|\mathbf{a}\|_1} \leq C_1^{\|\mathbf{a}\|_1} \sup_{t \in U} |s|^{\mathbf{a} \cdot \overline{M}}(t)$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $s \in H^0(\mathbf{a} \cdot \mathbf{M})$.

Proposition 2.3. Let $\overline{M} = (\overline{M}_1, \dots, \overline{M}_r)$ and \overline{E} be C^∞ -Hermitian line bundles on T . Fix an $s_0 \in H^0(E) \setminus \{0\}$. The \mathbb{C} -vector space $H^0(\mathbf{a} \cdot \mathbf{M})$ is endowed with the two norms $\|\cdot\|_{\sup}^{\mathbf{a} \cdot \overline{M}}$ and $\|\cdot\|_{\sup, \text{sub}(s_0^{\otimes b})}^{\mathbf{a} \cdot \overline{M} + b\overline{E}}$, where $\|\cdot\|_{\sup, \text{sub}(s_0^{\otimes b})}^{\mathbf{a} \cdot \overline{M} + b\overline{E}}$ is the subspace norm induced from $(H^0(\mathbf{a} \cdot \mathbf{M} + bE), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \overline{M} + b\overline{E}})$ via $H^0(\mathbf{a} \cdot \mathbf{M}) \xrightarrow{s_0^{\otimes b}} H^0(\mathbf{a} \cdot \mathbf{M} + bE)$. There then exists a constant $C_2 \geq 1$ depending only on \overline{M} , (\overline{E}, s_0) , and T such that

$$\|\cdot\|_{\sup}^{\mathbf{a} \cdot \overline{M}} \leq C_2^{\|\mathbf{a}\|_1 + b} \|\cdot\|_{\sup, \text{sub}(s_0^{\otimes b})}^{\mathbf{a} \cdot \overline{M} + b\overline{E}}$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$.

Proof. We choose a nonempty open subset U of T such that

$$\delta := \inf_{t \in U} |s_0|^{\overline{E}}(t) > 0$$

and such that $T_i \cap U \neq \emptyset$ for all i . By Lemma 2.2, there is a $C_1 \geq 1$ such that

$$\|s\|_{\sup}^{\mathbf{a} \cdot \overline{M}} \leq C_1^{\|\mathbf{a}\|_1} \sup_{t \in U} |s|^{\mathbf{a} \cdot \overline{M}}(t)$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $s \in H^0(\mathbf{a} \cdot \mathbf{M})$. Hence

$$\begin{aligned} \|s\|_{\sup}^{\mathbf{a} \cdot \overline{M}} &\leq \delta^{-b} C_1^{\|\mathbf{a}\|_1} \sup_{t \in U} |s \otimes s_0^{\otimes b}|^{\mathbf{a} \cdot \overline{M} + b\overline{E}}(t) \\ &\leq \max\{\delta^{-1}, C_1\}^{\|\mathbf{a}\|_1 + b} \|s \otimes s_0^{\otimes b}\|_{\sup}^{\mathbf{a} \cdot \overline{M} + b\overline{E}} \end{aligned}$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$, $b \in \mathbb{Z}_{\geq 0}$, and $s \in H^0(\mathbf{a} \cdot \mathbf{M})$. \square

2.2.2. Let T and Ω be as above, and consider the L^2 -norms with respect to Ω . Let $\overline{M} = (M, |\cdot|_{\overline{M}})$ be a C^∞ -Hermitian line bundle on T , let V be a linear series belonging to M , and let e_1, \dots, e_l be an L^2 -orthonormal basis for V . We define the *Bergman distortion function* $\beta(V; \overline{M}, \Omega)$ as

$$(2.11) \quad \beta(V; \overline{M}, \Omega)(x) := \sum_{i=1}^l |e_i|_{\overline{M}}(x)^2$$

for $x \in T$. It is easy to see that $\beta(V; \overline{M}, \Omega)$ does not depend on a specific choice of e_1, \dots, e_l . If $V = H^0(M)$, then we abbreviate

$$(2.12) \quad \beta(\overline{M}) := \beta(H^0(M); \overline{M}, \Omega)$$

for simplicity. The distortion function has the following elementary properties.

- (a) If W is a linear series containing V , then $\beta(V; \overline{M}, \Omega) \leq \beta(W; \overline{M}, \Omega)$.
- (b) For a $c \in \mathbb{R}_{>0}$, $\beta(V; \overline{M}, c\Omega) = c^{-1} \beta(V; \overline{M}, \Omega)$.

Proposition 2.4 ([12, Theorem 1.2.1]). *Let \overline{A} and $\overline{B} = (\overline{B}_1, \dots, \overline{B}_r)$ be C^∞ -Hermitian line bundles on T such that A and B are all ample and such that the Hermitian metrics are all pointwise positive definite. Suppose that the volume form is given as $\Omega := c_1(\overline{A})^{\wedge d}$. There then exists a constant $C_3 > 0$ such that*

$$\|\beta(a\overline{A} - b \cdot \overline{B})\|_{\sup} \leq C_3 a^d$$

for every $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}_{\geq 0}^r$.

2.2.3. Let \mathcal{X} be a projective arithmetic variety over $\text{Spec}(\mathbb{Z})$, let $\overline{\mathcal{M}}$ and $\overline{\mathcal{A}}$ be continuous Hermitian line bundles on \mathcal{X} , and fix an $s_0 \in H^0(\mathcal{A}) \setminus \{0\}$. The \mathbb{Z} -module $H^0(\mathcal{M}; \mathcal{V})$ is endowed with the supremum norm $\|\cdot\|_{\sup}$ and the L^2 -norm $\|\cdot\|_{L^2}$. Let $\|\cdot\|_{\sup, \text{sub}(s_0)}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}}$ (respectively, $\|\cdot\|_{L^2, \text{sub}(s_0)}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}}$) be the subspace norm induced via $H^0(\mathcal{M}; \mathcal{V}) \xrightarrow{\otimes s_0} H^0(\mathcal{M} + \mathcal{A})$; namely,

$$(2.13) \quad \|s\|_{\sup, \text{sub}(s_0)}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}} := \|s \otimes s_0\|_{\sup}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}} \quad (\text{respectively, } \|s\|_{L^2, \text{sub}(s_0)}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}} := \|s \otimes s_0\|_{L^2}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}})$$

for $s \in H^0(\mathcal{M}; \mathcal{V})$. For $*$ = s and ss, we write

$$(2.14) \quad \widehat{\ell}_{\text{sub}(s_0)}^* (\overline{\mathcal{M}}; \mathcal{V}) := \widehat{\ell}^* (H^0(\mathcal{M}; \mathcal{V}), \|\cdot\|_{\sup, \text{sub}(s_0)}^{\overline{\mathcal{M}} + \overline{\mathcal{A}}})$$

for short. The next one plays a key role in showing the main estimate in section 2.3.

Theorem 2.5. *Let \mathcal{X} be a projective arithmetic variety of dimension $d + 1$ over $\text{Spec}(\mathbb{Z})$. We assume that the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is smooth over $\text{Spec}(\mathbb{Q})$. Let $\overline{\mathcal{L}} = (\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_r)$ and $\overline{\mathcal{A}}$ be C^∞ -Hermitian line bundles on \mathcal{X} , and fix an $s_0 \in \widehat{\Gamma}^s(\overline{\mathcal{A}}) \setminus \{0\}$. If the Hermitian metrics of $\overline{\mathcal{L}}_1 + \overline{\mathcal{A}}, \dots, \overline{\mathcal{L}}_r + \overline{\mathcal{A}}$, and $\overline{\mathcal{A}}$ are all pointwise positive definite, then there exists a constant $C_4 > 0$ depending only on $\overline{\mathcal{L}}, (\overline{\mathcal{A}}, s_0)$, and \mathcal{X} such that*

$$\widehat{\ell}_{\text{sub}(s_0^{\otimes b})}^* (a \cdot \overline{\mathcal{L}}; \mathcal{V}) \leq \widehat{\ell}^* (a \cdot \overline{\mathcal{L}}; \mathcal{V}) + C_4 \|a\|_1^d (b + \log \|a\|_1)$$

for $*$ = s, ss, $a \in \mathbb{Z}_{\geq 0}^r$ with $\|a\|_1 > 0$, $b \in \mathbb{Z}_{\geq 0}$, and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$.

Proof. We set the volume form as

$$\Omega := c_1(\overline{\mathcal{L}}_1 + \dots + \overline{\mathcal{L}}_r + r\overline{\mathcal{A}})^{\wedge d},$$

and consider the L^2 -norms with respect to Ω . By Proposition 2.4, there exists a constant $D_1 > 0$ such that

$$\begin{aligned} & \left\| \beta \left(H^0(a \cdot \overline{\mathcal{L}}; \mathcal{V}); a \cdot \overline{\mathcal{L}}, \Omega \right) \right\|_{\sup} \\ & \leq \left\| \beta \left(\|a\|_1 (\overline{\mathcal{L}}_1 + \dots + \overline{\mathcal{L}}_r + r\overline{\mathcal{A}}) - \sum_{i=1}^r (\|a\|_1 - a_i) (\overline{\mathcal{L}}_i + \overline{\mathcal{A}}) - \|a\|_1 \overline{\mathcal{A}} \right) \right\|_{\sup} \\ & \leq D_1 \|a\|_1^d \end{aligned}$$

for every $a \in \mathbb{Z}_{\geq 0}^r$ with $\|a\|_1 > 0$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$.

We fix an L^2 -orthonormal basis e_1, \dots, e_l for $H^0(a \cdot \overline{\mathcal{L}}; \mathcal{V})$ in which the Hermitian form,

$$(s, t) \mapsto \langle s \otimes s_0^{\otimes b}, t \otimes s_0^{\otimes b} \rangle_{L^2}^{a \cdot \overline{\mathcal{L}} + b \cdot \overline{\mathcal{A}}},$$

is diagonalized. By [11, Corollary 1.1.2], we can change the supremum norms to the L^2 -norms up to error term $O(\|\mathbf{a}\|_1^d \log \|\mathbf{a}\|_1)$ (see [12, Proof of Lemma 1.3.3]). Since

$$\|\cdot\|_{L^2, \text{sub}(s_0^{\otimes b})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b \overline{\mathcal{A}}} \leq \|\cdot\|_{L^2}^{\mathbf{a} \cdot \overline{\mathcal{L}}} \cdot \left(\|s_0\|_{\text{sup}}^{\overline{\mathcal{A}}} \right)^b \leq \|\cdot\|_{L^2}^{\mathbf{a} \cdot \overline{\mathcal{L}}},$$

we can apply the inequality (1.5) and find a constant $D_2 > 0$ such that

$$\begin{aligned} (2.15) \quad & \widehat{\ell}_{\text{sub}(s_0^{\otimes b})}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) - \widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) - D_2 \|\mathbf{a}\|_1^d \log \|\mathbf{a}\|_1 \\ & \leq -\frac{1}{2} \log \left(\frac{\det \left(\langle e_i \otimes s_0^{\otimes b}, e_j \otimes s_0^{\otimes b} \rangle_{L^2}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b \overline{\mathcal{A}}} \right)_{i,j}}{\det \left(\langle e_i, e_j \rangle_{L^2}^{\mathbf{a} \cdot \overline{\mathcal{L}}} \right)_{i,j}} \right) \\ & = -\frac{1}{2} \sum_{i=1}^l \log \int_{\mathcal{X}(\mathbb{C})} \left(|e_i|^{\mathbf{a} \cdot \overline{\mathcal{L}}} \right)^2 \cdot \left(|s_0|^{\overline{\mathcal{A}}} \right)^{2b} \Omega \end{aligned}$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}$ with $\|\mathbf{a}\|_1 > 0$, $b \in \mathbb{Z}_{\geq 0}$, and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$.

Since $\int_{\mathcal{X}(\mathbb{C})} \left(|e_i|^{\mathbf{a} \cdot \overline{\mathcal{L}}} \right)^2 \Omega = 1$ for each i and s_0 does not vanish identically on each connected component of $\mathcal{X}(\mathbb{C})$, one can apply Jensen's inequality [15, Theorem 3.3] to the right-hand side of (2.15) and obtain

$$\begin{aligned} & \widehat{\ell}_{\text{sub}(s_0^{\otimes b})}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) - \widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) - D_2 \|\mathbf{a}\|_1^d \log \|\mathbf{a}\|_1 \\ & \leq \sum_{i=1}^l \int_{\mathcal{X}(\mathbb{C})} \left(|e_i|^{\mathbf{a} \cdot \overline{\mathcal{L}}} \right)^2 \cdot (-b \log |s_0|) \Omega \\ & \leq \left(D_1 \int_{\mathcal{X}(\mathbb{C})} -\log |s_0| \Omega \right) \cdot \|\mathbf{a}\|_1^d b. \end{aligned}$$

□

2.3. Main estimate: the case of models. Let \mathcal{X} be a projective arithmetic variety over $\text{Spec}(\mathbb{Z})$ of dimension $d+1$. Given a family $\overline{\mathcal{L}} := (\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_r)$ of continuous Hermitian line bundles on \mathcal{X} and an $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$, we write

$$\mathbf{a} \cdot \overline{\mathcal{L}} := a_1 \overline{\mathcal{L}}_1 + \dots + a_r \overline{\mathcal{L}}_r, \quad \mathbf{a} \cdot \mathcal{L} := a_1 \mathcal{L}_1 + \dots + a_r \mathcal{L}_r,$$

and $\|\mathbf{a}\|_1 := |a_1| + \dots + |a_r|$ as in Notation and terminology 1.1.1. The purpose of this section is to show the following estimate.

Theorem 2.6. *Let \mathcal{X} be a projective arithmetic variety of dimension $d+1$ over $\text{Spec}(\mathbb{Z})$. Assume that the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is smooth over $\text{Spec}(\mathbb{Q})$. Let $\overline{\mathcal{L}} := (\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_r)$ be a family of C^∞ -Hermitian line bundles on \mathcal{X} and let Σ be a finite set of points on \mathcal{X} . Let $\overline{\mathcal{A}}$ be any continuous Hermitian line bundle on \mathcal{X} . There then exists a constant $C > 0$ depending only on \mathcal{X} , $\overline{\mathcal{L}}$, $\overline{\mathcal{A}}$, and Σ such that*

$$\widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b \overline{\mathcal{A}}; \mathcal{V} \right) - \widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) \leq C \left((\|\mathbf{a}\|_1 + |b|)^d |b| + \|\mathbf{a}\|_1^d \log \|\mathbf{a}\|_1 \right)$$

for every $\mathbf{a} \in \mathbb{Z}^r$ with $\|\mathbf{a}\|_1 > 0$, $b \in \mathbb{Z}$, and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$ with

$$\{c_{\mathcal{X}}(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma.$$

Proof. We divide the proof into five steps.

Step 1. We may assume $\mathcal{V} \geq 0$. By considering $\pm \overline{\mathcal{L}}_1, \dots, \pm \overline{\mathcal{L}}_r$, and $\pm \overline{\mathcal{A}}$, one can observe that it suffices to show the theorem for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 > 0$ and $b \in \mathbb{Z}_{\geq 0}$. Moreover, if the theorem is true for an $\overline{\mathcal{A}}$, then it is also true for any $\overline{\mathcal{A}}'$ with $\overline{\mathcal{A}}' \leq \overline{\mathcal{A}}$ in place of $\overline{\mathcal{A}}$. Hence we can assume without loss of generality that $\overline{\mathcal{A}}$ has the following four properties.

- (a) \mathcal{A} is ample on \mathcal{X} .
- (b) The Hermitian metric of $\overline{\mathcal{A}}$ is C^∞ , and the Hermitian metrics of $\overline{\mathcal{L}}_1 + \overline{\mathcal{A}}, \dots, \overline{\mathcal{L}}_r + \overline{\mathcal{A}}$, and $\overline{\mathcal{A}}$ are all pointwise positive definite.
- (c) For every $n \gg 1$, $\langle \widehat{\Gamma}^{\text{ss}}(n\overline{\mathcal{A}}) \rangle_{\mathbb{Z}} = H^0(n\mathcal{A})$ (see Notation and terminology 1.1.1 and [8, Lemma 5.3]).
- (d) There is a nonzero small section $s_0 \in \widehat{\Gamma}^{\text{s}}(\overline{\mathcal{A}})$ such that $\text{div}(s_0)_{\mathbb{Q}}$ is smooth and such that s_0 does not pass through any point in Σ .

Step 2. For each $k \in \mathbb{Z}_{>0}$, we set

$$(2.16) \quad k\mathcal{Y} := \text{div}(s_0^{\otimes k}).$$

For $\mathbf{a} \in \mathbb{Z}^r$ and $b \in \mathbb{Z}$, we consider the \mathbb{Z} -module

$$(2.17) \quad H_{\mathcal{X}|k\mathcal{Y}}^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \\ := \text{Image} \left(H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \rightarrow H^0((\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{k\mathcal{Y}}) \right)$$

endowed with the quotient norm $\|\cdot\|_{\text{sup,quot}(\mathcal{X}|k\mathcal{Y})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}}$ induced from

$$\left(H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}), \|\cdot\|_{\text{sup}}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}} \right).$$

By abuse of notation, we will abbreviate, for $\bullet = \text{sub}(-)$, $\text{quot}(\mathcal{X}|-)$, etc.,

$$\widehat{\ell}_{\bullet}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) := \widehat{\ell}^* \left(H_{\bullet}^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}), \|\cdot\|_{\text{sup},\bullet}^? \right)$$

for simplicity, which in practice will cause no confusion.

By Snapper's theorem [9, page 295], one can find a constant $C > 0$ depending only on $\overline{\mathcal{L}}$, $\overline{\mathcal{A}}$, \mathcal{X} , and \mathcal{Y} such that

$$(2.18) \quad h^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}) \leq C(\|\mathbf{a}\|_1 + b)^d \quad \text{and} \quad h^0((\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{Y}}) \leq C(\|\mathbf{a}\|_1 + b)^{d-1}$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$ with $\|\mathbf{a}\|_1 + b > 0$.

In the rest of the proof, the constant C will be fittingly changed without explicit mentioning of it.

Claim 2.7. *There exists a constant $C > 0$ depending only on $\overline{\mathcal{L}}$, $\overline{\mathcal{A}}$, and \mathcal{Y} such that*

$$\widehat{\ell}_{\text{quot}(\mathcal{X}|\mathcal{Y})}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) \leq C(\|\mathbf{a}\|_1 + b)^d$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$ with $\|\mathbf{a}\|_1 + b > 0$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$.

Proof. It suffices to show the estimate

$$\widehat{\ell}^* \left((\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{Y}} \right) \leq C(\|\mathbf{a}\|_1 + b)^d$$

for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$ with $\|\mathbf{a}\|_1 + b > 0$. Let $\mathcal{Y}_{\text{horiz}}$ be the horizontal part of \mathcal{Y} , that is, the Zariski closure of $\mathcal{Y}_{\mathbb{Q}}$ in \mathcal{X} . Let \mathcal{I} (respectively, $\mathcal{I}_{\text{horiz}}$) be the ideal sheaf defining \mathcal{Y} (respectively, $\mathcal{Y}_{\text{horiz}}$) in \mathcal{X} . By the properties (a) and (c) of Step 1, one

finds an $n \in \mathbb{Z}_{>0}$ and nonzero small sections $t_i \in \widehat{\Gamma}^s(n\overline{\mathcal{A}} - \overline{\mathcal{L}}_i)$ for $i = 1, \dots, r$ such that each t_i does not pass through any associated point of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}_{\text{horiz}}$ and $\mathcal{I}_{\text{horiz}}/\mathcal{I}$.

First, one finds a constant $C > 0$ such that

$$(2.19) \quad \widehat{\ell}^* \left((a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{I}_{\text{horiz}}} \right) \leq \widehat{\ell}^* \left((n\|a\|_1 + b)\overline{\mathcal{A}}|_{\mathcal{I}_{\text{horiz}}} \right) \leq C(\|a\|_1 + b)^d$$

for every $a \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$ with $\|a\|_1 + b > 0$ (see for example [3, Theorem 2.8]).

Next, $\mathcal{I}_{\text{horiz}}/\mathcal{I}$ is a torsion sheaf having support of dimension $\leq d$. So, by Snapper's theorem, one has

$$(2.20) \quad \log \#H^0((a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}) \otimes \mathcal{I}_{\text{horiz}}/\mathcal{I}) \leq \log \#H^0((n\|a\|_1 + b)\overline{\mathcal{A}} \otimes \mathcal{I}_{\text{horiz}}/\mathcal{I}) \leq C(\|a\|_1 + b)^d$$

for every $a \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$ with $\|a\|_1 + b > 0$.

Applying (1.1) to the exact sequence

$$0 \rightarrow (a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}) \otimes (\mathcal{I}_{\text{horiz}}/\mathcal{I}) \rightarrow (a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{Y}} \rightarrow (a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{I}_{\text{horiz}}} \rightarrow 0,$$

one obtains, by (2.19) and (2.20),

$$\begin{aligned} & \widehat{\ell}^* \left((a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{Y}} \right) \\ & \leq \widehat{\ell}^* \left((a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}})|_{\mathcal{I}_{\text{horiz}}} \right) + \log \#H^0((a \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}) \otimes (\mathcal{I}_{\text{horiz}}/\mathcal{I})) \\ & \leq C(\|a\|_1 + b)^d \end{aligned}$$

for every $a \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{\geq 0}$ with $\|a\|_1 + b > 0$. \square

Step 3. We begin the estimation with the following claim.

Claim 2.8. *Let $k \in \mathbb{Z}_{>0}$, let \mathcal{M} be a line bundle on \mathcal{X} , and let $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$ such that $\{c_{\mathcal{X}}(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma$.*

- (1) *Tensoring by $s_0^{\otimes k}$ induces a homomorphism $H^0(\mathcal{M}; \mathcal{V}) \rightarrow H^0(\mathcal{M} + k\mathcal{A}; \mathcal{V})$.*
- (2) *The sequence*

$$0 \rightarrow H^0(\mathcal{M}; \mathcal{V}) \xrightarrow{\otimes s_0^{\otimes k}} H^0(\mathcal{M} + k\mathcal{A}; \mathcal{V}) \xrightarrow{q} H_{\mathcal{X}|_{k\mathcal{Y}}}^0(\mathcal{M} + k\mathcal{A}; \mathcal{V}) \rightarrow 0$$

is exact.

Proof. (1): Let $t \in H^0(\mathcal{M}; \mathcal{V})$. By the property (d) of Step 1, one has $\nu(s_0) = 0$ for every ν with $\nu(\mathcal{V}) > 0$. By (2.4),

$$\nu(t \otimes s_0^{\otimes k}) = \nu(t) + k\nu(s_0) \begin{cases} = \nu(t) \geq \nu(\mathcal{V}) & \text{if } \nu(\mathcal{V}) > 0 \text{ and} \\ \geq 0 & \text{if } \nu(\mathcal{V}) = 0 \end{cases}$$

for every $\nu \in \mathfrak{V}(\text{Rat}(\mathcal{X}))$; hence $t \otimes s_0^{\otimes k} \in H^0(\mathcal{M} + k\mathcal{A}; \mathcal{V})$.

(2): Suppose that $t \in H^0(\mathcal{M} + k\mathcal{A}; \mathcal{V})$ satisfies $q(t) = 0$; hence one finds a $t_0 \in H^0(\mathcal{M})$ such that $t = t_0 \otimes s_0^{\otimes k}$. By (2.4) and the property (d) of Step 1,

$$\nu(t_0) \begin{cases} = \nu(t) \geq \nu(\mathcal{V}) & \text{if } \nu(\mathcal{V}) > 0 \text{ and} \\ \geq 0 & \text{if } \nu(\mathcal{V}) = 0 \end{cases}$$

for every $\nu \in \mathfrak{V}(\text{Rat}(\mathcal{X}))$; hence $t_0 \in H^0(\mathcal{M}; \mathcal{V})$. \square

If $b = 0$, then the theorem is obvious, so that we can assume $b > 0$. We apply (1.1) to the exact sequence

$$(2.21) \quad 0 \rightarrow H^0(\mathbf{a} \cdot \mathcal{L}; \mathcal{V}) \xrightarrow{\otimes s_0^{\otimes b}} H^0(\mathbf{a} \cdot \mathcal{L} + b\mathcal{A}; \mathcal{V}) \rightarrow H^0_{\mathcal{X}|b\mathcal{Y}}(\mathbf{a} \cdot \mathcal{L} + b\mathcal{A}; \mathcal{V}) \rightarrow 0,$$

and obtain, by (2.18) and Theorem 2.5,

$$(2.22) \quad \begin{aligned} & \widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) \\ & \leq \widehat{\ell}^*_{\text{sub}(s_0^{\otimes b})} \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) + \widehat{\ell}^*_{\text{quot}(\mathcal{X}|b\mathcal{Y})} \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) \\ & \quad + C \|\mathbf{a}\|_1^d (1 + \log \|\mathbf{a}\|_1) \\ & \leq \widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) + \widehat{\ell}^*_{\text{quot}(\mathcal{X}|b\mathcal{Y})} \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) + C \|\mathbf{a}\|_1^d (b + \log \|\mathbf{a}\|_1) \end{aligned}$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 > 0$ and $b \in \mathbb{Z}_{> 0}$.

Step 4. We are going to estimate the middle term $\widehat{\ell}^*_{\text{quot}(\mathcal{X}|b\mathcal{Y})} \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right)$ in the right-hand side of (2.22). For each $k \in \mathbb{Z}_{> 0}$, we identify $-k\mathcal{A}$ with an ideal sheaf of $\mathcal{O}_{\mathcal{X}}$ via the morphism $-k\mathcal{A} \xrightarrow{\otimes s_0^{\otimes k}} \mathcal{O}_{\mathcal{X}}$. The inclusions $-(k+1)\mathcal{A} \xrightarrow{\otimes s_0} -k\mathcal{A} \xrightarrow{\otimes s_0^{\otimes k}} \mathcal{O}_{\mathcal{X}}$ induce an injective morphism

$$\begin{aligned} \sigma_k : -k\mathcal{A}|_{\mathcal{Y}} &= \text{Coker} \left(-(k+1)\mathcal{A} \xrightarrow{\otimes s_0} -k\mathcal{A} \right) \\ &\rightarrow \text{Coker} \left(-(k+1)\mathcal{A} \xrightarrow{\otimes s_0^{\otimes (k+1)}} \mathcal{O}_{\mathcal{X}} \right) = \mathcal{O}_{(k+1)\mathcal{Y}}. \end{aligned}$$

Claim 2.9. *For each $k \in \mathbb{Z}_{> 0}$, σ_k induces a homomorphism*

$$H^0_{\mathcal{X}|\mathcal{Y}}(\mathbf{a} \cdot \mathcal{L} + (b-k)\mathcal{A}; \mathcal{V}) \rightarrow H^0_{\mathcal{X}|(k+1)\mathcal{Y}}(\mathbf{a} \cdot \mathcal{L} + b\mathcal{A}; \mathcal{V}).$$

Proof. This is obvious because σ_k induces a homomorphism

$$H^0((\mathbf{a} \cdot \mathcal{L} + (b-k)\mathcal{A})|_{\mathcal{Y}}) \rightarrow H^0((\mathbf{a} \cdot \mathcal{L} + b\mathcal{A})|_{(k+1)\mathcal{Y}})$$

and the diagram

$$\begin{array}{ccc} H^0((\mathbf{a} \cdot \mathcal{L} + (b-k)\mathcal{A})|_{\mathcal{Y}}) & \xrightarrow{\sigma_k} & H^0((\mathbf{a} \cdot \mathcal{L} + b\mathcal{A})|_{(k+1)\mathcal{Y}}) \\ \uparrow & & \uparrow \\ H^0(\mathbf{a} \cdot \mathcal{L} + (b-k)\mathcal{A}; \mathcal{V}) & \xrightarrow{\otimes s_0^{\otimes k}} & H^0(\mathbf{a} \cdot \mathcal{L} + b\mathcal{A}; \mathcal{V}) \end{array}$$

is commutative. \square

A commutative diagram of $\mathcal{O}_{\mathcal{X}}$ -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & -k\mathcal{A}|_{\mathcal{Y}} & \xrightarrow{\sigma_k} & \mathcal{O}_{(k+1)\mathcal{Y}} & \longrightarrow & \mathcal{O}_{k\mathcal{Y}} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & -k\mathcal{A} & \xrightarrow{\otimes s_0^{\otimes k}} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_{k\mathcal{Y}} \longrightarrow 0, \end{array}$$

yields a commutative diagram of \mathbb{Z} -modules:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0_{\mathcal{X}|\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) & \xrightarrow{\sigma_k} & H^0_{\mathcal{X}|(k+1)\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) & \longrightarrow & H^0_{\mathcal{X}|k\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \parallel \\
0 & \longrightarrow & H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) & \xrightarrow{\otimes s_0^{\otimes k}} & H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) & \longrightarrow & H^0_{\mathcal{X}|k\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \longrightarrow 0.
\end{array}$$

Since the right vertical arrow is an identity and the lower horizontal sequence is exact (see Claim 2.8), one sees that the upper horizontal sequence of the diagram is also exact. Applying (1.1) to the upper horizontal sequence, one obtains

$$\begin{aligned}
(2.23) \quad & \widehat{\ell}^*_{\text{quot}(\mathcal{X}|(k+1)\mathcal{Y})}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \\
& \leq \widehat{\ell}^*_{\text{quot}(\mathcal{X}|(k+1)\mathcal{Y}), \text{sub}(\sigma_k)}(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) \\
& \quad + \widehat{\ell}^*_{\text{quot}(\mathcal{X}|k\mathcal{Y})}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) + C(\|\mathbf{a}\|_1 + b)^{d-1} (1 + \log(\|\mathbf{a}\|_1 + b))
\end{aligned}$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, k \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 + b > 0$ and $k \leq b$ (see (2.18)).

Step 5. By applying [11, Lemma 3.4(2)] to the right square of the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k-1)\overline{\mathcal{A}}; \mathcal{V}) & \xrightarrow{\otimes s_0^{\otimes (k+1)}} & H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) & \longrightarrow & H^0_{\mathcal{X}|(k+1)\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \longrightarrow 0 \\
& & \parallel & & \uparrow \otimes s_0^{\otimes k} & & \uparrow \sigma_k \\
0 & \longrightarrow & H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k-1)\overline{\mathcal{A}}; \mathcal{V}) & \xrightarrow{\otimes s_0} & H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) & \longrightarrow & H^0_{\mathcal{X}|k\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) \longrightarrow 0,
\end{array}$$

and by using Proposition 2.3, one can get a constant D with $0 < D \leq 1$ such that

$$\begin{aligned}
\|\cdot\|_{\text{sup}, \text{quot}(\mathcal{X}|(k+1)\mathcal{Y}), \text{sub}(\sigma_k)}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}} &= \|\cdot\|_{\text{sup}, \text{sub}(s_0^{\otimes k}), \text{quot}(\mathcal{X}|\mathcal{Y})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}} \\
&\geq D\|\mathbf{a}\|_1 + b \|\cdot\|_{\text{sup}, \text{quot}(\mathcal{X}|\mathcal{Y})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}}
\end{aligned}$$

on $H^0_{\mathcal{X}|\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V})$, where $\|\cdot\|_{\text{sup}, \text{quot}(\mathcal{X}|(k+1)\mathcal{Y}), \text{sub}(\sigma_k)}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}}$ is the subspace norm induced from

$$\left(H^0_{\mathcal{X}|(k+1)\mathcal{Y}}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}), \|\cdot\|_{\text{sup}, \text{quot}(\mathcal{X}|(k+1)\mathcal{Y})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}} \right)$$

via σ_k , and $\|\cdot\|_{\text{sup}, \text{sub}(s_0^{\otimes k}), \text{quot}(\mathcal{X}|\mathcal{Y})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}}$ is the quotient norm induced from

$$\left(H^0(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}), \|\cdot\|_{\text{sup}, \text{sub}(s_0^{\otimes k})}^{\mathbf{a} \cdot \overline{\mathcal{L}} + \overline{\mathcal{A}}} \right).$$

Hence, by (1.2), (1.4), (2.18), and Claim 2.7, one gets a constant $C > 0$ such that

$$\begin{aligned}
(2.24) \quad & \widehat{\ell}^*_{\text{quot}(\mathcal{X}|(k+1)\mathcal{Y}), \text{sub}(\sigma_k)}(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) \\
& \leq \widehat{\ell}^*_{\text{quot}(\mathcal{X}|\mathcal{Y})}(\mathbf{a} \cdot \overline{\mathcal{L}} + (b-k)\overline{\mathcal{A}}; \mathcal{V}) + C(\|\mathbf{a}\|_1 + b)^d \leq C(\|\mathbf{a}\|_1 + b)^d
\end{aligned}$$

for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b, k \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 + b > 0$ and $k \leq b$.

By (2.23) and (2.24),

$$\begin{aligned}
(2.25) \quad & \widehat{\ell}^*_{\text{quot}(\mathcal{X}|(k+1)\mathcal{Y})}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \leq \widehat{\ell}^*_{\text{quot}(\mathcal{X}|k\mathcal{Y})}(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V}) \\
& \quad + C(\|\mathbf{a}\|_1 + b)^d.
\end{aligned}$$

By summing up (2.25) for $k = 1, 2, \dots, b-1$ and by using Claim 2.7 again, one has

$$\begin{aligned} \widehat{\ell}_{\text{quot}(\mathcal{X}|b\mathcal{Y})}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) &\leq \widehat{\ell}_{\text{quot}(\mathcal{X}|\mathcal{Y})}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) + C(\|\mathbf{a}\|_1 + b)^d b \\ &\leq C(\|\mathbf{a}\|_1 + b)^d b. \end{aligned}$$

Therefore, by (2.22), one obtains

$$\widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}} + b\overline{\mathcal{A}}; \mathcal{V} \right) \leq \widehat{\ell}^* \left(\mathbf{a} \cdot \overline{\mathcal{L}}; \mathcal{V} \right) + C((\|\mathbf{a}\|_1 + b)^d b + \|\mathbf{a}\|_1^d \log \|\mathbf{a}\|_1)$$

for every $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 > 0$ and $b \in \mathbb{Z}_{>0}$. \square

3. ARITHMETIC VOLUMES OF ℓ^1 -ADELIC \mathbb{R} -CARTIER DIVISORS

3.1. Preliminaries. In this section, we recall definitions and basic properties of adelically normed vector spaces (section 3.1.1), Berkovich analytic spaces (section 3.1.2), and adelic Green functions (section 3.1.3).

3.1.1. Let K be a number field. Let $\overline{V} := (V, (\|\cdot\|_v^{\overline{V}})_{v \in M_K})$ be a couple of a finite-dimensional K -vector space V and a collection $(\|\cdot\|_v^{\overline{V}})_{v \in M_K}$ such that each $\|\cdot\|_v^{\overline{V}}$ is a $(K_v, |\cdot|_v)$ -norm on $V_{K_v} = V \otimes_K K_v$ and such that, if $v \in M_K^{\text{fin}}$, then $\|\cdot\|_v^{\overline{V}}$ is non-Archimedean. For such a \overline{V} , we set

$$(3.1) \quad \widehat{\Gamma}^f(\overline{V}) := \left\{ s \in V : \|s\|_v^{\overline{V}} \leq 1 \text{ for every } v \in M_K^{\text{fin}} \right\},$$

$$(3.2) \quad \widehat{\Gamma}^s(\overline{V}) := \left\{ s \in \widehat{\Gamma}^f(\overline{V}) : \|s\|_{\infty}^{\overline{V}} \leq 1 \right\}, \quad \widehat{\Gamma}^{\text{ss}}(\overline{V}) := \left\{ s \in \widehat{\Gamma}^s(\overline{V}) : \|s\|_{\infty}^{\overline{V}} < 1 \right\},$$

and $\widehat{\ell}^*(\overline{V}) := \log \# \widehat{\Gamma}^*(\overline{V})$ for $*$ = s and ss. Note that $\widehat{\Gamma}^f(\overline{V})$ is a O_K -submodule of V and $\widehat{\ell}^*(\overline{V})$ may be infinite.

We set, for $\lambda \in \mathbb{R}$,

$$\mathcal{F}^{\lambda}(\overline{V}) := \left\langle s \in \widehat{\Gamma}^f(\overline{V}) : \|s\|_{\infty}^{\overline{V}} \leq e^{-\lambda} \right\rangle_K$$

(see Notation and terminology 1.1.1), and set

$$(3.3) \quad e_{\max}(\overline{V}) := \sup \{ \lambda \in \mathbb{R} : \mathcal{F}^{\lambda}(\overline{V}) \neq 0 \}.$$

Proposition 3.1. *Let $\overline{V} = (V, (\|\cdot\|_v^{\overline{V}})_{v \in M_K})$ be a couple of a finite-dimensional K -vector space V and a collection $(\|\cdot\|_v^{\overline{V}})_{v \in M_K}$ such that each $\|\cdot\|_v^{\overline{V}}$ is a $(K_v, |\cdot|_v)$ -norm on V_{K_v} and such that, if $v \in M_K^{\text{fin}}$, then $\|\cdot\|_v^{\overline{V}}$ is non-Archimedean.*

(1) *The following are equivalent.*

- (a) *For each $s \in V$, $\|s\|_v^{\overline{V}} \leq 1$ for all but finitely many $v \in M_K$.*
- (b) *$\widehat{\Gamma}^f(\overline{V})$ contains an O_K -submodule E of V satisfying $E_K = V$.*

(2) *Suppose that \overline{V} satisfies the equivalent conditions of (1). The following are then equivalent.*

- (a) *$\widehat{\Gamma}^f(\overline{V})$ is a finitely generated O_K -module.*
- (b) *$\widehat{\Gamma}^s(\overline{V})$ is finite.*
- (c) *$\widehat{\Gamma}^{\text{ss}}(\overline{V})$ is finite.*
- (d) *$e_{\max}(\overline{V}) < +\infty$.*

Proof. (1) (a) \Rightarrow (b): For each $s \in V$, one can find an $n \geq 1$ such that $\|ns\|_v^\overline{V} = |n|_v \|s\|_v^\overline{V} \leq 1$ for every $v \in M_K$ by the condition (a). Thus $ns \in \widehat{\Gamma}^f(\overline{V})$, which implies $V = \widehat{\Gamma}^f(\overline{V})_K$.

(b) \Rightarrow (a): For each $s \in V$, there exists an $\alpha \in O_K$ such that $\alpha s \in E$. Hence $\|\alpha s\|_v^\overline{V} = \|\alpha\|_v^\overline{V} \|s\|_v^\overline{V} \leq 1$ for all but finitely many $v \in M_K$.

For the assertion (2), we refer to [3, Proposition 2.4] and [2, Proposition C.2.4]. \square

Definition 3.1. An *adelically normed K -vector space* is a couple $(V, (\|\cdot\|_v^\overline{V})_{v \in M_K})$ satisfying the all conditions in Proposition 3.1(1),(2). Notice that here the existence of an O_K -model of \overline{V} that defines $\|\cdot\|_v^\overline{V}$ except for finitely many v is not assumed while it is in the classical definition in [17, (1.6)] and in [4, Definition 3.1].

Let $\lambda \in \mathbb{R}$ and let $v \in M_K$. We define an adelicly normed K -vector space $\overline{V}(\lambda[v]) = (V, (\|\cdot\|_w^{\overline{V}(\lambda[v])})_{w \in M_K})$ as

$$(3.4) \quad \|\cdot\|_w^{\overline{V}(\lambda[v])} := \begin{cases} \|\cdot\|_w^\overline{V} & \text{if } w \neq v \text{ and} \\ e^{-\lambda} \|\cdot\|_v^\overline{V} & \text{if } w = v. \end{cases}$$

Lemma 3.2. Let $\lambda \in \mathbb{R}_{\geq 0}$ and let $v \in M_K^{\text{fin}}$. If we set $p\mathbb{Z} := \mathfrak{p}_v \cap \mathbb{Z}$, then

$$0 \leq \widehat{\ell}^*(\overline{V}(\lambda[v])) - \widehat{\ell}^*(\overline{V}) \leq \left(\left\lceil \frac{\lambda}{-\log |p|_v} \right\rceil \log(p) + 2 \right) \dim_{\mathbb{Q}} V.$$

Proof. Set

$$n_\lambda := \left\lceil \frac{\lambda}{-\log |p|_v} \right\rceil.$$

We are going to show

$$(3.5) \quad p^{n_\lambda} \widehat{\Gamma}^f(\overline{V}(\lambda[v])) \subset \widehat{\Gamma}^f(\overline{V}).$$

Suppose that $s \in \widehat{\Gamma}^f(\overline{V}(\lambda[v]))$. Then

$$\|p^{n_\lambda} s\|_w^\overline{V} = |p|_w^{n_\lambda} \|s\|_w^\overline{V} \leq \begin{cases} 1 & \text{if } w \neq v \text{ and} \\ e^{\lambda} |p|_v^{n_\lambda} & \text{if } w = v. \end{cases}$$

Since $\lambda + n_\lambda \log |p|_v \leq 0$, we have $p^{n_\lambda} s \in \widehat{\Gamma}^f(\overline{V})$.

We apply (1.3) to the inclusion $\widehat{\Gamma}^f(\overline{V}) \subset \widehat{\Gamma}^f(\overline{V}(\lambda[v]))$, and obtain

$$\begin{aligned} \widehat{\ell}^*(\overline{V}(\lambda[v])) &\leq \widehat{\ell}^*(\overline{V}) + \log \# \left(\widehat{\Gamma}^f(\overline{V}(\lambda[v])) / \widehat{\Gamma}^f(\overline{V}) \right) + 2 \dim_{\mathbb{Q}} V \\ &\leq \widehat{\ell}^*(\overline{V}) + \log \# \left(\widehat{\Gamma}^f(\overline{V}(\lambda[v])) / p^{n_\lambda} \widehat{\Gamma}^f(\overline{V}(\lambda[v])) \right) + 2 \dim_{\mathbb{Q}} V \\ &\leq \widehat{\ell}^*(\overline{V}) + (n_\lambda \log(p) + 2) \dim_{\mathbb{Q}} V \end{aligned}$$

by (3.5). \square

3.1.2. Let X be a normal, projective, and geometrically connected K -variety. For $v = \infty$, we denote by X_∞^{an} the complex analytic space associated to $X_{\mathbb{C}} := X \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{C})$. For $v \in M_K^{\text{fin}}$, we denote by $(X_v^{\text{an}}, \rho_v : X_v^{\text{an}} \rightarrow X_{K_v})$ the Berkovich analytic space associated to X_{K_v} (see [1]). For each $x \in X_v^{\text{an}}$, we denote

by $\kappa(x)$ the residue field of $\rho_v(x) \in X_{K_v}$ and by $|\cdot|_x$ the corresponding norm on $\kappa(x)$. Given a local function f on X_{K_v} defined around $\rho_v(x)$, we write

$$(3.6) \quad |f|(x) := |f(\rho_v(x))|_x.$$

An O_K -model of X is a reduced, irreducible, projective, and flat O_K -scheme with generic fiber $\mathcal{X}_K \simeq X$. Given an O_K -model \mathcal{X} of X , we set

$$(3.7) \quad \widetilde{\mathcal{X}}_v := \mathcal{X} \times_{\text{Spec}(O_K)} \text{Spec}(\widetilde{K}_v).$$

For each $x \in X_v^{\text{an}}$, the morphism $\rho_v(x) : \text{Spec}(\kappa(x)) \rightarrow \mathcal{X}_{K_v^\circ}$ uniquely extends to a morphism $\text{Spec}(\kappa(x)^\circ) \rightarrow \mathcal{X}_{K_v^\circ}$ by the valuative criterion of properness. We define $r_v^{\mathcal{X}}(x)$ as the image of the closed point of $\text{Spec}(\kappa(x)^\circ)$.

Let $\mathcal{U} = \text{Spec}(\mathcal{A})$ be an affine open subscheme of $\mathcal{X}_{K_v^\circ}$ with $\mathcal{U} \cap \widetilde{\mathcal{X}}_v \neq \emptyset$, and set $U = \mathcal{U}_{K_v} = \text{Spec}(A)$. We put

$$(3.8) \quad U_{v,\mathcal{U}}^{\text{an}} := \{x \in U_v^{\text{an}} : |f|(x) \leq 1 \text{ for all } f \in \mathcal{A}\}.$$

Lemma 3.3. (1) $U_{v,\mathcal{U}}^{\text{an}} = (r_v^{\mathcal{X}})^{-1}(\mathcal{U} \cap \widetilde{\mathcal{X}}_v)$.
 (2) $U_{v,\mathcal{U}}^{\text{an}}$ is compact.

Proof. (1): If $x \in U_{v,\mathcal{U}}^{\text{an}}$, then the image of the homomorphism $\mathcal{A} \rightarrow \kappa(x)$ is in $\kappa(x)^\circ$, so $r_v^{\mathcal{X}}(x) \in \mathcal{U}$. Conversely, if $r_v^{\mathcal{X}}(x) \in \mathcal{U} \cap \widetilde{\mathcal{X}}_v$, then $\rho_v(x) \in U$ and $x \in \rho_v^{-1}(U) = U_v^{\text{an}}$. Since $r_v^{\mathcal{X}}(x) \in \mathcal{U}$, the image of the morphism $\text{Spec}(\kappa(x)^\circ) \rightarrow \mathcal{X}_{K_v^\circ}$ is in \mathcal{U} , so $f(\rho_v(x)) \in \kappa(x)^\circ$ for every $f \in \mathcal{A}$.

(2): The map

$$u : U_v^{\text{an}} \rightarrow I := \prod_{f \in \mathcal{A}} \mathbb{R}_{\geq 0}, \quad x \mapsto (|f|(x))_{f \in \mathcal{A}},$$

is injective and continuous, where I is endowed with the product topology. By Tychonoff's theorem, $J := \prod_{f \in \mathcal{A}} [0, 1]$ is a compact subset of I , and $U_{v,\mathcal{U}}^{\text{an}} = u^{-1}(J)$. Thus it suffices to show that u is a closed map. Suppose that $(u(x_\alpha))_\alpha$ is a net in I that converges to $(\lambda_f)_{f \in \mathcal{A}} \in I$. For each $f \in \mathcal{A}$, we set $|f|_x := \lambda_f$.

Claim 3.4. $|\cdot|_x$ extends to a multiplicative seminorm on A whose restriction to K_v is $|\cdot|_v$.

Proof of Claim 3.4. Since, for every α , $|\cdot|_{x_\alpha}$ satisfies the conditions:

- $|a|(x_\alpha) = |a|_v$ for $a \in K_v$,
- $|f - g|(x_\alpha) \leq |f|(x_\alpha) + |g|(x_\alpha)$ for $f, g \in \mathcal{A}$, and
- $|fg|(x_\alpha) = |f|(x_\alpha)|g|(x_\alpha)$ for $f, g \in \mathcal{A}$,

we know that the limit $|\cdot|_x$ is a multiplicative seminorm on \mathcal{A} . For a general $f \in A$, we can take an $n \geq 0$ such that $\varpi_v^n f \in \mathcal{A}$, and define

$$|f|_x := |\varpi_v|_v^{-n} |\varpi_v^n f|_x,$$

which does not depend on a specific choice of $n \geq 0$. Then $|\cdot|_x$ is a multiplicative seminorm on A . \square

By Claim 3.4, $|\cdot|_x$ corresponds to a point $x \in U_v^{\text{an}}$. Since $|f|_{x_\alpha} \rightarrow |f|_x$ for every $f \in A$, the net $(x_\alpha)_\alpha$ converges to x in the Gel'fand topology, and $(\lambda_f)_{f \in \mathcal{A}} = u(x)$. It implies that u is a closed map. \square

Let $\widetilde{\mathcal{X}}_{v,\text{gen}}$ be the set of all the generic points of irreducible components of $\widetilde{\mathcal{X}}_v$. For each $\xi \in \widetilde{\mathcal{X}}_{v,\text{gen}}$, $(r_v^{\mathcal{X}})^{-1}(\xi)$ consists of a single point x_ξ given as

$$(3.9) \quad |\phi|_{x_\xi} := \left(\# \widetilde{K}_v \right)^{-\frac{\text{ord}_\xi(\phi)}{\text{ord}_\xi(\varpi_v)}}$$

for $\phi \in \text{Rat}(X)$. We set $\Gamma(X_v^{\text{an}}) := \{x_\xi : \xi \in \widetilde{\mathcal{X}}_{v,\text{gen}}\}$ (see also [1, Proposition 2.4.4 and Corollary 2.4.5]).

Lemma 3.5. *Suppose that \mathcal{A} is integrally closed in A . Then, for each $f \in A$,*

$$\max_{x \in U_{v,\mathcal{U}}^{\text{an}}} \{|f|(x)\} = \max_{x \in \Gamma(X_v^{\text{an}}) \cap U_{v,\mathcal{U}}^{\text{an}}} \{|f|(x)\}.$$

Proof. Since $\mathcal{U} \cap \widetilde{\mathcal{X}}_v \neq \emptyset$, we have $\Gamma(X_v^{\text{an}}) \cap U_{v,\mathcal{U}}^{\text{an}} \neq \emptyset$. The inequality \geq is obvious, so that we are going to show the reverse. Choose a $\xi_0 \in \widetilde{\mathcal{X}}_{v,\text{gen}}$ such that

$$|f|(x_{\xi_0}) = \max_{x \in \Gamma(X_v^{\text{an}}) \cap U_{v,\mathcal{U}}^{\text{an}}} \{|f|(x)\}.$$

If we set $n := \text{ord}_{\xi_0}(\varpi_v)$ and $l := \text{ord}_{\xi_0}(f)$, then $\text{ord}_\xi(\varpi_v^{-l} f^n) \geq 0$ for every $\xi \in \widetilde{\mathcal{X}}_{v,\text{gen}}$. By [8, Lemma 2.3(3)], it implies $\varpi_v^{-l} f^n \in \mathcal{A}$. Hence

$$|f|(x) \leq |\varpi_v|_v^{\frac{l}{n}} = |f|(x_{\xi_0})$$

for every $x \in U_{v,\mathcal{U}}^{\text{an}}$. □

3.1.3. Let X be a normal, projective, and geometrically connected K -variety, let $\mathbb{K} = \mathbb{R}, \mathbb{Q}$, or \mathbb{Z} , and let D be a \mathbb{K} -Cartier divisor on X . The *support* of D is a Zariski closed subset defined as

$$(3.10) \quad \text{Supp}(D) := \bigcup_{\substack{Z: \text{ prime Weil divisor,} \\ \text{ord}_Z(D) \neq 0}} Z$$

(see [7, Notation and terminology 2]). Let $v \in M_K$. A *D -Green function* on X_v^{an} is a continuous map $g_v : (X \setminus \text{Supp}(D))_v^{\text{an}} \rightarrow \mathbb{R}$ such that, for each $x \in X_v^{\text{an}}$,

$$(3.11) \quad g_v(x) + \log |f|^2(x)$$

extends to a continuous function around x , where $f \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{K}$ is a local equation defining D around $\rho_v(x)$ (see [14, Definition 2.1.1]). If $v = \infty$, we assume that a D -Green function is invariant under the complex conjugation map. We then set

$$(3.12) \quad \widehat{\text{Div}}_{\mathbb{K}}^{\text{tot}}(X) := \left\{ \left(D, \sum_{v \in M_K} g_v^{\overline{D}}[v] \right) : \begin{array}{l} D \in \text{Div}_{\mathbb{K}}(X) \text{ and } g_v^{\overline{D}} \text{ is a } D\text{-Green} \\ \text{function on } X_v^{\text{an}} \text{ for each } v \in M_K \end{array} \right\}.$$

An element $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X)$ is called *effective* if

$$(3.13) \quad D \geq 0 \quad \text{and} \quad \text{ess.inf}_{x \in X_v^{\text{an}}} g_v^{\overline{D}}(x) \geq 0, \quad \forall v \in M_K,$$

and, for $\overline{D}, \overline{E} \in \widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X)$, we write $\overline{D} \leq \overline{E}$ if $\overline{E} - \overline{D}$ is effective. Each $g_v^{\overline{D}}$ defines the supremum norm on $H^0(D)$ as

$$(3.14) \quad \|\phi\|_{v,\text{sup}}^{\overline{D}} := \sup_{x \in X_v^{\text{an}}} |\phi|(x) \exp \left(\frac{1}{2} g_v^{\overline{D}}(x) \right)$$

for $\phi \in H^0(D)$ (see [14, Proposition 2.1.3]).

In the following, we impose on $\nu \in \mathfrak{V}(\text{Rat}(X))$ a condition that the restriction of ν to K is trivial (see section 2.1). Given a $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X)$ and a $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$, we set

$$(3.15) \quad \widehat{\Gamma}^*(\overline{D}; \mathcal{V}) := \widehat{\Gamma}^*\left(H^0(D; \mathcal{V}), (\|\cdot\|_{v, \text{sup}})_{v \in M_K}\right)$$

for $*$ = f, s, and ss, and set $\widehat{\ell}^*(\overline{D}; \mathcal{V}) := \log \# \widehat{\Gamma}^*(\overline{D}; \mathcal{V})$ for $*$ = s and ss (see section 3.1.1 and (2.10)). An O_K -model of a couple (X, D) is a couple $(\mathcal{X}, \mathcal{D})$ such that \mathcal{X} is a normal O_K -model of X and such that \mathcal{D} is an \mathbb{R} -Cartier divisor on \mathcal{X} satisfying $\mathcal{D}|_{\mathcal{X}} = D$. Given an O_K -model $(\mathcal{X}, \mathcal{D})$ of (X, D) and a $v \in M_K^{\text{fin}}$, we define the D -Green function associated to $(\mathcal{X}, \mathcal{D})$ as

$$(3.16) \quad g_v^{(\mathcal{X}, \mathcal{D})}(x) := -\log |f'|^2(x),$$

where f' is a local equation defining \mathcal{D} around $r_v^{\mathcal{X}}(x)$.

Let $\mathbb{K} := \mathbb{R}, \mathbb{Q}$, or \mathbb{Z} . A couple $\overline{\mathcal{D}} = (\mathcal{D}, g_{\infty}^{\overline{\mathcal{D}}})$ on \mathcal{X} such that $(\mathcal{X}, \mathcal{D})$ is an O_K -model of (X, D) with $\mathcal{D} \in \text{Div}_{\mathbb{K}}(\mathcal{X})$ and such that $g_{\infty}^{\overline{\mathcal{D}}}$ is a D -Green function on X_{∞}^{an} is called an *arithmetic \mathbb{K} -Cartier divisor* on \mathcal{X} . If X is smooth and $g_{\infty}^{\overline{\mathcal{D}}}$ is of C^{∞} -type, then $\overline{\mathcal{D}}$ is said to be of C^{∞} -type (see [13, section 2.3]). We denote by $\widehat{\text{Div}}_{\mathbb{K}}(\mathcal{X})$ (respectively, $\widehat{\text{Div}}_{\mathbb{K}}(\mathcal{X}; C^{\infty})$) the \mathbb{K} -module of all the arithmetic \mathbb{K} -Cartier divisors (respectively, arithmetic \mathbb{K} -Cartier divisors of C^{∞} -type) on \mathcal{X} . If $\mathbb{K} = \mathbb{Z}$ and $-$ = a blank or C^{∞} , we will abbreviate $\widehat{\text{Div}}(\mathcal{X}; -) := \widehat{\text{Div}}_{\mathbb{Z}}(\mathcal{X}; -)$ as usual.

Given a couple $(\overline{\mathcal{D}}; \mathcal{V})$ of a $\overline{\mathcal{D}} \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X})$ and a $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$, we abbreviate

$$(3.17) \quad \widehat{\ell}^*(\overline{\mathcal{D}}; \mathcal{V}) := \widehat{\ell}^*\left(H^0(\overline{\mathcal{D}}; \mathcal{V}), \|\cdot\|_{\infty, \text{sup}}\right)$$

for $*$ = s and ss (see Notation and terminology 1.1.2 and (2.10)), and define

$$(3.18) \quad \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}) := \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s(m\overline{\mathcal{D}}; m\mathcal{V})}{m^{\dim \mathcal{X}} / (\dim \mathcal{X})!}.$$

Moreover, the *adelization* of $\overline{\mathcal{D}} \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X})$ is defined as

$$(3.19) \quad \overline{\mathcal{D}}^{\text{ad}} := \left(D, \sum_{v \in M_K^{\text{fin}}} g_v^{(\mathcal{X}, \mathcal{D})}[v] + g_{\infty}[\infty] \right),$$

which belongs to $\widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X)$.

3.2. The space of continuous functions. Let K be a number field, and let X be a projective and geometrically connected K -variety. For each $v \in M_K$, we denote by $C(X_v^{\text{an}})$ the space of \mathbb{R} -valued continuous functions on X_v^{an} that are assumed to be invariant under the complex conjugation if $v = \infty$. We endow $C(X_v^{\text{an}})$ with the supremum norm:

$$\|f\|_{\text{sup}} := \sup_{x \in X_v^{\text{an}}} |f(x)|_{\infty}$$

for $f \in C(X_v^{\text{an}})$, where $|f(x)|_{\infty}$ denotes the usual absolute value of the real number $f(x)$ (see Notation and terminology 1.1.4). By elementary arguments, $(C(X_v^{\text{an}}), \|\cdot\|_{\text{sup}})$

$\|\cdot\|_{\text{sup}}$) is a Banach algebra for every $v \in M_K$. We denote by

$$(3.20) \quad C_{\text{tot}}(X) := \prod_{v \in M_K} C(X_v^{\text{an}}) = \left\{ \mathbf{f} = \sum_{v \in M_K} f_v[v] : f_v \in C(X_v^{\text{an}}) \right\}$$

the algebraic direct product of the family $(C(X_v^{\text{an}}))_{v \in M_K}$, and by

$$(3.21) \quad C(X) := \bigoplus_{v \in M_K} C(X_v^{\text{an}})$$

the algebraic direct sum of $(C(X_v^{\text{an}}))_{v \in M_K}$. The ℓ^1 -norm of an $\mathbf{f} \in C_{\text{tot}}(X)$ is

$$(3.22) \quad \|\mathbf{f}\|_{\ell^1} := \sum_{v \in M_K} \|f_v\|_{\text{sup}},$$

where the sum is taken with respect to the net indexed by all the finite subsets of M_K , and the ℓ^1 -direct sum of $(C(X_v^{\text{an}}))_{v \in M_K}$ is given as

$$C_{\ell^1}(X) := \{ \mathbf{f} = (f_v)_{v \in M_K} : \|\mathbf{f}\|_{\ell^1} < +\infty \}$$

endowed with the ℓ^1 -norm. For $\mathbf{f}, \mathbf{g} \in C_{\text{tot}}(X)$, we write $\mathbf{f} \leq \mathbf{g}$ if $f_v \leq g_v$ for every $v \in M_K$. If $\mathbf{f}, \mathbf{g} \in C_{\ell^1}(X)$, then the entrywise product $\mathbf{f}\mathbf{g}$ satisfies

$$\|\mathbf{f}\mathbf{g}\|_{\ell^1} \leq \sum_{v \in M_K} \|f_v\|_{\text{sup}} \|g_v\|_{\text{sup}} \leq \sup_{v \in M_K} \{\|f_v\|_{\text{sup}}\} \cdot \|\mathbf{g}\|_{\ell^1} \leq \|\mathbf{f}\|_{\ell^1} \cdot \|\mathbf{g}\|_{\ell^1},$$

so $\mathbf{f}\mathbf{g} \in C_{\ell^1}(X)$. By the same arguments as in [15, page 67, Theorem 3.11], one verifies that $(C_{\ell^1}(X), \|\cdot\|_{\ell^1})$ is a Banach algebra. Note that $C_{\text{tot}}(\text{Spec}(K))$ is just \mathbb{R}^{M_K} and $C_{\ell^1}(\text{Spec}(K)) = \ell^1(M_K)$ is just the ℓ^1 -sequence space indexed by M_K . We will identify $C_{\text{tot}}(\text{Spec}(K))$ with the space of constant functions in $C_{\text{tot}}(X)$.

Lemma 3.6. *Let $\mathbf{f} \in C_{\ell^1}(X)$. Given any $\varepsilon > 0$, there exists a $\mathbf{h} \in C(X)$ such that*

$$\mathbf{h} \leq \mathbf{f} \quad \text{and} \quad \|\mathbf{f} - \mathbf{h}\|_{\ell^1} \leq \varepsilon.$$

Proof. Since $\sum_{v \in M_K} \|f_v\|_{\text{sup}} < +\infty$, there is a finite subset $S \subset M_K$ such that

$$\sum_{v \in (M_K \setminus S)} \|f_v\|_{\text{sup}} \leq \varepsilon.$$

Hence $\mathbf{h} := \sum_{v \in S} f_v[v]$ satisfies the required conditions. \square

3.3. ℓ^1 -adelic \mathbb{R} -Cartier divisors. Let X be a normal, projective, and geometrically connected K -variety. The natural homomorphisms

$$(3.23) \quad C_{\text{tot}}(X) \rightarrow \widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X), \quad \mathbf{f} \mapsto (0, \mathbf{f}),$$

and

$$(3.24) \quad \zeta : \widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X) \rightarrow \text{Div}_{\mathbb{R}}(X), \quad \overline{D} = \left(D, \sum_{v \in M_K} g_v^{\overline{D}}[v] \right) \mapsto \zeta(\overline{D}) = D,$$

form an exact sequence

$$(3.25) \quad 0 \rightarrow C_{\text{tot}}(X) \rightarrow \widehat{\text{Div}}_{\mathbb{R}}^{\text{tot}}(X) \xrightarrow{\zeta} \text{Div}_{\mathbb{R}}(X) \rightarrow 0.$$

Let \mathbb{K} and \mathbb{K}' be either \mathbb{R} , \mathbb{Q} , or \mathbb{Z} . Given a $\overline{D} \in \widehat{\text{Div}}_{\mathbb{K}}^{\text{tot}}(X)$, we set

$$(3.26) \quad \widehat{\text{Mod}}_{\mathbb{K}'}(\overline{D}) := \left\{ (\mathcal{X}, (\mathcal{D}, g_{\infty})) : \begin{array}{l} (\mathcal{X}, \mathcal{D}) \text{ is an } O_K\text{-model of } (X, D), (\mathcal{D}, g_{\infty}) \\ \in \widehat{\text{Div}}_{\mathbb{K}'}(\mathcal{X}), \text{ and } \overline{\mathcal{D}}^{\text{ad}} \leq \overline{D} \end{array} \right\}.$$

We call $\overline{D} \in \widehat{\text{Div}}_{\mathbb{K}}^{\text{tot}}(X)$ an *adelic \mathbb{K} -Cartier divisor* if there exists an $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}_{\mathbb{R}}(\overline{D})$ such that $\overline{D} - \overline{\mathcal{D}}^{\text{ad}} \in C(X)$. Denote by $\widehat{\text{Div}}_{\mathbb{K}}(X)$ the \mathbb{K} -module of all the adelic \mathbb{K} -Cartier divisors on X . As before, we will write $\widehat{\text{Div}}(X) := \widehat{\text{Div}}_{\mathbb{Z}}(X)$. For a $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, there are a nonempty open subset U of $\text{Spec}(O_K)$ and an $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}_{\mathbb{R}}(\overline{D})$ such that $g_v^{\overline{D}} = g_v^{(\mathcal{X}, \overline{\mathcal{D}})}$ for every $v \in U$. In this case, we call the couple $(\mathcal{X}_U, \overline{\mathcal{D}}_U)$ a *U -model of definition* for \overline{D} (see [14, Definition 4.1.1] and [7, Notation and terminology 4]). Given a $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ and a $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$, we define

$$(3.27) \quad \widehat{\text{vol}}(\overline{D}; \mathcal{V}) := \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!},$$

which is finite as in [7, section 2.5].

Proposition 3.7. *Let $\mathbb{K} = \mathbb{R}$ or \mathbb{Q} . For any $\overline{D} \in \widehat{\text{Div}}_{\mathbb{K}}^{\text{tot}}(X)$, the following are equivalent.*

- (1) *There exists an $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}_{\mathbb{K}}(\overline{D})$ such that $\|\overline{D} - \overline{\mathcal{D}}^{\text{ad}}\|_{\ell^1} < +\infty$.*
- (2) *For any $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}_{\mathbb{R}}(\overline{D})$, $\|\overline{D} - \overline{\mathcal{D}}^{\text{ad}}\|_{\ell^1} < +\infty$.*
- (3) *For any $\varepsilon > 0$, there exists an $(\mathcal{X}_{\varepsilon}, \overline{\mathcal{D}}_{\varepsilon}) \in \widehat{\text{Mod}}_{\mathbb{K}}(\overline{D})$ such that $\|\overline{D} - \overline{\mathcal{D}}_{\varepsilon}^{\text{ad}}\|_{\ell^1} \leq \varepsilon$.*
- (4) *For any $\varepsilon > 0$, there exists an $\overline{D}_{\varepsilon} \in \widehat{\text{Div}}_{\mathbb{K}}(X)$ such that $\zeta(\overline{D}_{\varepsilon}) = \zeta(\overline{D})$, $\overline{D}_{\varepsilon} \leq \overline{D}$, and $\|\overline{D} - \overline{D}_{\varepsilon}\|_{\ell^1} \leq \varepsilon$.*

Proof. The implications (2) \Rightarrow (1) and (3) \Rightarrow (1) are obvious. The equivalence (3) \Leftrightarrow (4) results from the approximation theorem (see [14, Theorem 4.1.3]).

(1) \Rightarrow (2): It suffices to show that for any $(\mathcal{X}', \overline{\mathcal{D}}') \in \widehat{\text{Mod}}_{\mathbb{R}}(\overline{D})$

$$(3.28) \quad \|\overline{\mathcal{D}}^{\text{ad}} - \overline{\mathcal{D}}'^{\text{ad}}\|_{\ell^1} < +\infty.$$

Let \mathcal{X}'' be a normal O_K -model of X that dominates both \mathcal{X} and \mathcal{X}' . Let $\mu : \mathcal{X}'' \rightarrow \mathcal{X}$ and $\mu' : \mathcal{X}'' \rightarrow \mathcal{X}'$ be the dominant morphisms. Then

$$(\mu^* \overline{\mathcal{D}})^{\text{ad}} = \overline{\mathcal{D}}^{\text{ad}} \quad \text{and} \quad (\mu'^* \overline{\mathcal{D}}')^{\text{ad}} = \overline{\mathcal{D}}'^{\text{ad}}$$

(see [14, Proposition 2.1.4]). We write

$$D = a_1 Z_1 + \cdots + a_r Z_r$$

with $a_i \in \mathbb{R}$ and prime Weil divisors Z_i . Let \mathcal{Z}_i be the Zariski closure of Z_i in \mathcal{X}'' . Since

$$\mu^* \overline{\mathcal{D}} - \sum_{i=1}^r a_i \mathcal{Z}_i \quad \text{and} \quad \mu'^* \overline{\mathcal{D}}' - \sum_{i=1}^r a_i \mathcal{Z}_i$$

are both vertical, one can find a nonempty open subset $U \subset \text{Spec}(O_K)$ such that $(\mu^* \overline{\mathcal{D}})_U = (\mu'^* \overline{\mathcal{D}}')_U$. Hence we have (3.28).

(1) \Rightarrow (4): Set $(0, \mathbf{f}) := \overline{D} - \overline{\mathcal{D}}^{\text{ad}}$. By Lemma 3.6, there exists an $\mathbf{f}_{\varepsilon} \in C(X)$ such that $\mathbf{f}_{\varepsilon} \leq \mathbf{f}$ and such that $\|\mathbf{f} - \mathbf{f}_{\varepsilon}\|_{\ell^1} \leq \varepsilon$. Set

$$\overline{D}_{\varepsilon} := \overline{\mathcal{D}}^{\text{ad}} + (0, \mathbf{f}_{\varepsilon}).$$

Then $\overline{D}_{\varepsilon} \in \widehat{\text{Div}}_{\mathbb{K}}(X)$, $\overline{D}_{\varepsilon} \leq \overline{D}$, and $\|\overline{D} - \overline{D}_{\varepsilon}\|_{\ell^1} = \|\mathbf{f} - \mathbf{f}_{\varepsilon}\|_{\ell^1} \leq \varepsilon$. \square

Definition 3.2. Let $\mathbb{K} = \mathbb{R}, \mathbb{Q}, \text{ or } \mathbb{Z}$. We call an element $\overline{D} \in \widehat{\text{Div}}_{\mathbb{K}}^{\text{tot}}(X)$ an ℓ^1 -adelic \mathbb{K} -Cartier divisor on X if there exists an $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}_{\mathbb{R}}(\overline{D})$ such that $\|\overline{D} - \overline{\mathcal{D}}^{\text{ad}}\|_{\ell^1} < +\infty$. We denote by $\widehat{\text{Div}}_{\mathbb{K}}^{\ell^1}(X)$ the \mathbb{K} -module of all the ℓ^1 -adelic \mathbb{K} -Cartier divisors on X . If $\mathbb{K} = \mathbb{Z}$, then the subscript \mathbb{Z} will be omitted as usual.

Moreover, we set

$$(3.29) \quad \widehat{\text{Div}}_{\mathbb{K}, \mathbb{R}}^{\ell^1}(X) := \widehat{\text{Div}}_{\mathbb{K}}^{\ell^1}(X) \times \text{BC}_{\mathbb{R}}(X).$$

Let $\mathbf{Pic}_{X/K}$ be the Picard scheme of X and let $\mathbf{Pic}_{X/K}^0$ be the neutral component of $\mathbf{Pic}_{X/K}$. Let $\text{Pic}(X) = \mathbf{Pic}_{X/K}(K)$ be the Picard group of X , and let

$$(3.30) \quad \text{NS}(X) := \mathbf{Pic}_{X/K}(\overline{K}) / \mathbf{Pic}_{X/K}^0(\overline{K})$$

be the Néron–Severi group of X . By Severi’s theorem of the base, $\text{NS}(X)$ is a finitely generated \mathbb{Z} -module, and, since $\mathbf{Pic}_{X/K}^0$ is an abelian variety over K (see for example [10, Theorem 5.4]), $\mathbf{Pic}_{X/K}^0(K)$ is also a finitely generated \mathbb{Z} -module by the Mordell–Weil theorem. Since

$$\mathbf{Pic}_{X/K}^0(\overline{K}) \cap \mathbf{Pic}_{X/K}(K) = \mathbf{Pic}_{X/K}^0(K),$$

we obtain an exact sequence

$$(3.31) \quad 0 \rightarrow \mathbf{Pic}_{X/K}^0(K) \rightarrow \text{Pic}(X) \rightarrow \text{NS}(X).$$

Hence $\text{Pic}(X)$ is also a finitely generated \mathbb{Z} -module.

Let $\widehat{P}_{\mathbb{R}}(X)$ (respectively, $P_{\mathbb{R}}(X)$) be the \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ (respectively, $\text{Div}_{\mathbb{R}}(X)$) generated by the principal divisors $(\widehat{\phi})$ (respectively, (ϕ)) for $\phi \in \text{Rat}(X)^{\times}$. Let $\text{Pic}_{\mathbb{R}}(X) := \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the \mathbb{R} -vector space of \mathbb{R} -line bundles on X . By [6, Proposition II.6.15], the sequence

$$(3.32) \quad 0 \rightarrow P_{\mathbb{R}}(X) \rightarrow \text{Div}_{\mathbb{R}}(X) \xrightarrow{\Theta_X} \text{Pic}_{\mathbb{R}}(X) \rightarrow 0$$

is exact. So, if we set

$$(3.33) \quad \text{Cl}_{\mathbb{R}}(X) := \text{Div}_{\mathbb{R}}(X) / P_{\mathbb{R}}(X),$$

then $\text{Cl}_{\mathbb{R}}(X) = \text{Pic}_{\mathbb{R}}(X)$ is a finite-dimensional \mathbb{R} -vector space.

Definition 3.3. We define

$$\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X) := \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X) / \widehat{P}_{\mathbb{R}}(X).$$

Lemma 3.8. *The sequence*

$$0 \rightarrow C_{\ell^1}(X) \rightarrow \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X) \xrightarrow{\zeta} \text{Cl}_{\mathbb{R}}(X) \rightarrow 0$$

is exact.

Proof. Obviously, the sequence

$$0 \rightarrow C_{\ell^1}(X) \rightarrow \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X) \xrightarrow{\zeta} \text{Div}_{\mathbb{R}}(X) \rightarrow 0$$

is exact. If $\zeta(\overline{D}) \in P_{\mathbb{R}}(X)$, then $D = (\phi)$ for a $\phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ or $D = 0$. Hence $\zeta^{-1}(P_{\mathbb{R}}(X)) = \widehat{P}_{\mathbb{R}}(X) \oplus C_{\ell^1}(X)$, which infers the required result. \square

We fix a section $\iota : \text{Cl}_{\mathbb{R}}(X) \rightarrow \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ of ζ and a norm $\|\cdot\|$ on the finite-dimensional \mathbb{R} -vector space $\text{Cl}_{\mathbb{R}}(X)$. We can then define a norm on $\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ as

$$(3.34) \quad \|\overline{D}\|_{\iota, \|\cdot\|} := \|D\| + \|\overline{D} - \iota(D)\|_{\ell^1}$$

for $\overline{D} \in \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$, where we regard $\overline{D} - \iota(D) \in C_{\ell^1}(X)$.

Proposition 3.9. *Let $\iota : \text{Cl}_{\mathbb{R}}(X) \rightarrow \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ be a section of ζ and let $\|\cdot\|$ be a norm on $\text{Cl}_{\mathbb{R}}(X)$.*

- (1) *$(\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X), \|\cdot\|_{\iota, \|\cdot\|})$ is a Banach space.*
- (2) *Let $\iota' : \text{Cl}_{\mathbb{R}}(X) \rightarrow \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ be another section and let $\|\cdot\|'$ be another norm. Then $\|\cdot\|_{\iota', \|\cdot\|'}$ is equivalent to $\|\cdot\|_{\iota, \|\cdot\|}$.*

Proof. (1): If $(\overline{D}_n)_{n \geq 1}$ is a Cauchy sequence in $\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$, then $(\zeta(\overline{D}_n))_{n \geq 1}$ is a Cauchy sequence in $\text{Cl}_{\mathbb{R}}(X)$, and converges to an $E \in \text{Cl}_{\mathbb{R}}(X)$. Set $(0, \mathbf{f}_n) := \overline{D}_n - \iota(\zeta(\overline{D}_n))$. The sequence $(\mathbf{f}_n)_{n \geq 1}$ is then a Cauchy sequence in $C_{\ell^1}(X)$, and converges to a $\mathbf{g} \in C_{\ell^1}(X)$. The sequence $(\overline{D}_n)_{n \geq 1}$ then converges to $\iota(E) + (0, \mathbf{g})$.

(2): It suffices to show $\|\cdot\|_{\iota', \|\cdot\|'} \leq C \|\cdot\|_{\iota, \|\cdot\|}$ for a $C > 0$. We choose a basis A_1, \dots, A_l for $\text{Cl}_{\mathbb{R}}(X)$ and set

$$\|a_1 A_1 + \dots + a_l A_l\|_1 := |a_1| + \dots + |a_l|.$$

We can find a constant $C_1 \geq 1$ such that $\|\cdot\|' \leq C_1 \|\cdot\|$ and such that $\|\cdot\|_1 \leq C_1 \|\cdot\|$. We set $(0, \mathbf{f}_i) := \iota(A_i) - \iota'(A_i)$ for each i , and set

$$C_2 := \max_{1 \leq i \leq l} \{\|\iota(A_i) - \iota'(A_i)\|_{\ell^1}, 1\}.$$

Then, for any $\overline{D} \in \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ with $D = a_1 A_1 + \dots + a_l A_l$,

$$\begin{aligned} \|\overline{D}\|_{\iota', \|\cdot\|'} &= \|D\|' + \|\overline{D} - \iota'(D)\|_{\ell^1} \\ &\leq C_1 \|D\| + \|\overline{D} - \iota(D)\|_{\ell^1} + \sum_{i=1}^l |a_i| \|\iota(A_i) - \iota'(A_i)\|_{\ell^1} \\ &\leq C_1 \|D\| + \|\overline{D} - \iota(D)\|_{\ell^1} + C_2 \|\overline{D}\|_1 \leq 2C_1 C_2 \|\overline{D}\|_{\iota, \|\cdot\|}. \end{aligned}$$

□

3.4. Arithmetic volume function. The following is a key idea to introduce the notion of ℓ^1 -adelic \mathbb{R} -Cartier divisors.

Lemma 3.10. *Let \mathcal{X} be a normal, projective, and geometrically connected arithmetic variety over $\text{Spec}(O_K)$, and let $\overline{\mathcal{D}} \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X})$. Suppose that every irreducible component of \mathcal{D} is Cartier. Let $U = U_{(\mathcal{X}, \mathcal{D})}$ be a nonempty open subset of $\text{Spec}(O_K)$ having the following properties.*

- (a) $\pi_U : \mathcal{X}_U \rightarrow U$ is geometrically reduced and geometrically irreducible.
- (b) For every $v \in U$, $\text{ord}_{\pi_U^{-1}(v)}(\mathcal{D}) = 0$.

Then, for every $v \in U$ and $\phi \in H^0(D) \setminus \{0\}$, one has

$$\inf_{x \in X_v^{\text{an}}} \left\{ g_v^{(\mathcal{X}, \mathcal{D})}(x) - \log |\phi|^2(x) \right\} \in (2 \log \# \tilde{K}_v) \mathbb{Z}.$$

Proof. By assumption, every irreducible component of $\mathcal{D}|_{\mathcal{X}_{K_v^\circ}}$ is Cartier, so we can write

$$\mathcal{D}|_{\mathcal{X}_{K_v^\circ}} = a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$$

with $a_i \in \mathbb{R}$ and prime Cartier divisors \mathcal{D}_i .

We choose a finite affine open covering $(\mathcal{U}_\lambda)_\lambda$ of $\mathcal{X}_{K_v^\circ}$ such that $\mathcal{U}_\lambda \cap \widetilde{\mathcal{X}}_v \neq \emptyset$ and $\mathcal{D}_i \cap \mathcal{U}_\lambda$ is principal with equation $f_{i,\lambda}$ for each λ . We set $\mathcal{U}_\lambda = \text{Spec}(\mathcal{A}_\lambda)$ with finitely generated and integrally closed K_v° -algebra \mathcal{A}_λ , and set $U_\lambda := \text{Spec}(\mathcal{A}_\lambda \otimes_{K_v^\circ} K)$. We then have

$$X_v^{\text{an}} = \bigcup_{\lambda} (U_\lambda)_{v, \mathcal{U}_\lambda}^{\text{an}} \quad \text{and} \quad \psi_\lambda := \phi \cdot f_{1,\lambda}^{\lfloor a_1 \rfloor} \cdots f_{r,\lambda}^{\lfloor a_r \rfloor} \in \mathcal{A}_\lambda$$

for every $\phi \in H^0(D) \setminus \{0\}$ and λ .

By Lemma 3.5, the function

$$(U_\lambda)_{v, \mathcal{U}_\lambda}^{\text{an}} \rightarrow \mathbb{R}, \quad x \mapsto |\psi_\lambda|(x) \cdot |f_{1,\lambda}|^{a_1 - \lfloor a_1 \rfloor}(x) \cdots |f_{r,\lambda}|^{a_r - \lfloor a_r \rfloor}(x),$$

attains its maximum at the single point in $\Gamma(X_v^{\text{an}}) \cap (U_\lambda)_{v, \mathcal{U}_\lambda}^{\text{an}}$ that corresponds to the fiber $\widetilde{\mathcal{X}}_v$. Let ϖ_v be a uniformizer of K_v . Since

$$\text{ord}_{\widetilde{\mathcal{X}}_v}(\varpi_v) = 1 \quad \text{and} \quad \text{ord}_{\widetilde{\mathcal{X}}_v}(f_{1,\lambda}) = \cdots = \text{ord}_{\widetilde{\mathcal{X}}_v}(f_{r,\lambda}) = 0,$$

we have

$$\begin{aligned} \frac{1}{2} \inf_{x \in (U_\lambda)_{v, \mathcal{U}_\lambda}^{\text{an}}} \left\{ g_v^{(\mathcal{X}, \mathcal{D})}(x) - \log |\phi|(x) \right\} &= \frac{\text{ord}_{\widetilde{\mathcal{X}}_v}(\phi \cdot f_{1,\lambda}^{a_1} \cdots f_{r,\lambda}^{a_r})}{\text{ord}_{\widetilde{\mathcal{X}}_v}(\varpi_v)} \cdot \log \# \widetilde{K}_v \\ &= \text{ord}_{\widetilde{\mathcal{X}}_v}(\phi) \log \# \widetilde{K}_v \in (\log \# \widetilde{K}_v) \mathbb{Z} \end{aligned}$$

for every λ . We have thus proved the lemma. \square

Proposition 3.11. *Let X be a normal, projective, and geometrically connected K -variety and let $\mu : \widetilde{X} \rightarrow X$ be a resolution of singularities of X . Let $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, and let $\mathbf{a} = \sum_{v \in M_K} a_v[v] \in C_{\text{tot}}(\text{Spec}(K))$ with $\mathbf{a} \geq 0$.*

(I) *Let U be a nonempty open subset of $\text{Spec}(O_K)$ over which a model of definition for \overline{D} exists.*

We choose an O_K -model $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ of $(\widetilde{X}, \mu^ D)$ such that $(\widetilde{\mathcal{X}}_U, \widetilde{\mathcal{D}}_U)$ gives a U -model of definition for $\mu^* \overline{D}$ and such that every irreducible component of $\widetilde{\mathcal{D}}$ is Cartier.*

(II) *Let $U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$ be a nonempty open subset of U such that $\pi : \widetilde{\mathcal{X}}_{U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}} \rightarrow U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$ is smooth and such that $\text{ord}_{\pi^{-1}(v)}(\widetilde{\mathcal{D}}) = 0$ for every $v \in U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$.*

(III) $U_{\mathbf{a}} := \left\{ v \in M_K^{\text{fin}} : a_v < 2 \log \# \widetilde{K}_v \right\}$.

We set

$$\mathbf{a}' := \sum_{v \notin U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})} \cap U_{\mathbf{a}}} a_v[v].$$

Then the following holds.

- (1) *If a $\mathbf{b} \in C_{\text{tot}}(\text{Spec}(K))$ satisfies $\mathbf{b} \geq \mathbf{a}$, then $U_{\mathbf{b}} \subset U_{\mathbf{a}}$.*
- (2) *For any $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$, one has*

$$\widehat{\Gamma}^f(\overline{D} + (0, \mathbf{a}); \mathcal{V}) = \widehat{\Gamma}^f(\overline{D} + (0, \mathbf{a}'); \mathcal{V}).$$

- (3) If $\sharp(M_K^{\text{fin}} \setminus U_{\mathbf{a}})$ is finite (in particular, if \mathbf{a} is a bounded sequence), then $\widehat{\ell}^*(\overline{D} + (0, \mathbf{a}); \mathcal{V})$ is finite for every $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ and $*$ = s, ss.

Proof. The assertion (1) is obvious.

(2): Since $\mathbf{a} \geq 0$, the inclusion \supset is obvious. Suppose $v \in U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})} \cap U_{\mathbf{a}}$; hence, in particular,

$$(3.35) \quad 0 \geq -a_v > -2 \log \sharp \widetilde{K}_v.$$

If $\phi \in H^0(\mu^* D; \mathcal{V}) \setminus \{0\} = H^0(D; \mathcal{V}) \setminus \{0\}$ satisfies

$$g_v^{\overline{D}}(x) + a_v - \log |\phi|^2(x) \geq 0$$

for every $x \in X_v^{\text{an}}$, then

$$g_v^{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}(x') - \log |\phi|^2(x') \geq -a_v$$

for every $x' \in \widetilde{X}_v^{\text{an}}$. By Lemma 3.10 and (3.35), we have

$$\inf_{x' \in \widetilde{X}_v^{\text{an}}} \left\{ g_v^{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}(x') - \log |\phi|^2(x') \right\} = \inf_{x \in X_v^{\text{an}}} \left\{ g_v^{\overline{D}}(x) - \log |\phi|^2(x) \right\} \geq 0.$$

Hence $\phi \in \widehat{\Gamma}^{\text{f}}(\overline{D} + (0, \mathbf{a}); \mathcal{V})$ implies $\phi \in \widehat{\Gamma}^{\text{f}}(\overline{D} + (0, \mathbf{a}'); \mathcal{V})$.

If $M_K \setminus U_{\mathbf{a}}$ is finite, then so is $M_K \setminus (U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})} \cap U_{\mathbf{a}})$. Hence, the assertion (2) implies the assertion (3) (see [14, Proposition 4.3.1(3)]). \square

Proposition 3.12. *Let X be a normal, projective, and geometrically connected K -variety and let $*$ = s or ss.*

- (1) *To each $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}(X)$, one can assign a constant $\delta(\overline{D}) > 0$, which depends only on \overline{D} and X , such that*

$$0 \leq \widehat{\ell}^*(\overline{D} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\ell}^*(\overline{D}; \mathcal{V}) \leq \left(\frac{3}{2} \|\mathbf{f}\|_{\ell^1} + \delta(\overline{D}) \right) \dim_{\mathbb{Q}} H^0(D; \mathcal{V}).$$

for every $\mathbf{f} \in C_{\ell^1}(X)$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$. Moreover, one can assume that

$$\delta(t\overline{D}) = \delta(\overline{D})$$

holds for every $t \in \mathbb{R} \setminus \{0\}$.

- (2) *For any $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$, $\widehat{\Gamma}^*(\overline{D}; \mathcal{V})$ is a finite set.*

- (3) *For any $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$, $(H^0(D; \mathcal{V}), (\|\cdot\|_{v, \text{sup}})_{v \in M_K})$ is an adelicly normed K -vector space.*

Proof. (1): Set

$$(3.36) \quad \mathbf{a} := \sum_{v \in M_K} \|f_v\|_{\text{sup}}[v] \in C_{\ell^1}(\text{Spec}(K)).$$

For each $v \in M_K^{\text{fin}}$, we denote by p_v the prime number satisfying $p_v \mathbb{Z} = \mathfrak{p}_v \cap \mathbb{Z}$.

Let $\mu : \widetilde{X} \rightarrow X$ be a resolution of singularities of X and let U be a nonempty open subset of $\text{Spec}(O_K)$ over which a model of definition for \overline{D} exists. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ be an O_K -model of $(\widetilde{X}, \mu^* D)$ such that $(\widetilde{\mathcal{X}}_U, \widetilde{\mathcal{D}}_U)$ gives a U -model of definition for $\mu^* \overline{D}$ and such that every irreducible component of $\widetilde{\mathcal{D}}$ is Cartier.

We choose the two nonempty open subsets $U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$ and $U_{\mathbf{a}}$ as in Proposition 3.11; namely,

- $U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$ is chosen to satisfy that $U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})} \subset U$, that $\pi : \widetilde{\mathcal{X}}_{U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}} \rightarrow U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$ is smooth, and that $\text{ord}_{\pi^{-1}(v)}(\widetilde{\mathcal{D}}) = 0$ for every $v \in U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$, and
- $U_{\mathbf{a}} := \left\{ v \in M_K^{\text{fin}} : a_v < 2 \log \# \widetilde{K}_v \right\}$.

We divide M_K^{fin} into three disjoint subsets: $S_1 := U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})} \cap U_{\mathbf{a}}$,

$$S_2 := \left\{ v \in M_K^{\text{fin}} : 2 \log \# \widetilde{K}_v \leq a_v \text{ and } 2 \leq \log(p_v) \right\},$$

and $S_3 := M_K^{\text{fin}} \setminus (S_1 \cup S_2)$. Note that only S_1 is an infinite subset and S_3 is contained in a finite subset

$$(3.37) \quad S'_3 := \left(M_K \setminus U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})} \right) \cup \{ v \in M_K^{\text{fin}} : \log(p_v) < 2 \},$$

which is determined only by $U_{(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})}$. Put

$$(3.38) \quad \mathbf{a}' := \sum_{v \in S_2} a_v[v] + \sum_{v \in S_3} a_v[v].$$

By Proposition 3.11(2), Lemma 3.2, and (1.2), we have

$$(3.39) \quad \begin{aligned} & \widehat{\ell}^* (\overline{D} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\ell}^* (\overline{D}; \mathcal{V}) \\ & \leq \widehat{\ell}^* (\overline{D} + (0, \mathbf{a}); \mathcal{V}) - \widehat{\ell}^* (\overline{D}; \mathcal{V}) \\ & = \widehat{\ell}^* (\overline{D} + (0, \mathbf{a}'); \mathcal{V}) - \widehat{\ell}^* (\overline{D}; \mathcal{V}) \\ & \leq \left(\sum_{v \in S_2 \cup S_3} \left(\left\lceil \frac{\|f_v\|_{\text{sup}}}{-2 \log |p_v|_v} \right\rceil \log(p_v) + 2 \right) + \frac{\|f_{\infty}\|_{\text{sup}}}{2} + 2 \right) \dim_{\mathbb{Q}} H^0(D; \mathcal{V}). \end{aligned}$$

We can estimate the sum with respect to S_2 as

$$(3.40) \quad \begin{aligned} & \sum_{v \in S_2} \left(\left\lceil \frac{\|f_v\|_{\text{sup}}}{-2 \log |p_v|_v} \right\rceil \log(p_v) + 2 \right) \\ & \leq \sum_{v \in S_2} \left(\frac{\|f_v\|_{\text{sup}}}{2 \text{ord}_v(p_v)[\widetilde{K}_v : \mathbb{F}_{p_v}]} + 2 \log(p_v) \right) \\ & \leq \sum_{v \in S_2} \left(\frac{\|f_v\|_{\text{sup}}}{2 \text{ord}_v(p_v)[\widetilde{K}_v : \mathbb{F}_{p_v}]} + \frac{\|f_v\|_{\text{sup}}}{[\widetilde{K}_v : \mathbb{F}_{p_v}]} \right) \leq \frac{3}{2} \sum_{v \in S_2} \|f_v\|_{\text{sup}} \end{aligned}$$

and the sum with respect to S_3 as

$$(3.41) \quad \sum_{v \in S_3} \left(\left\lceil \frac{\|f_v\|_{\text{sup}}}{-2 \log |p_v|_v} \right\rceil \log(p_v) + 2 \right) \leq \sum_{v \in S_3} \left(\frac{1}{2} \|f_v\|_{\text{sup}} + \log(p_v) + 2 \right).$$

Hence, if we set $p_{\infty} := 1$ and

$$(3.42) \quad \delta(\overline{D}) := \sum_{v \in S'_3} (\log(p_v) + 2),$$

then we obtain

$$\widehat{\ell}^* (\overline{D} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\ell}^* (\overline{D}; \mathcal{V}) \leq \left(\frac{3}{2} \|\mathbf{f}\|_{\ell^1} + \delta(\overline{D}) \right) \dim_{\mathbb{Q}} H^0(D; \mathcal{V})$$

by (3.39), (3.40), and (3.41). Since the constant $\delta(\overline{D})$ depends only on $U_{(\mathcal{X}, \overline{\mathcal{D}})}$, we have $\delta(t\overline{D}) = \delta(\overline{D})$ for every $t \in \mathbb{R} \setminus \{0\}$.

The assertion (2) is obvious from the assertion (1). The assertion (3) follows from the assertion (2) and the fact that $\widehat{\Gamma}^f(\overline{D}; \mathcal{V})$ contains $H^0(\mathcal{D}; \mathcal{V})$ for any $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}(\overline{D})$. \square

Proposition 3.13. *Let $*$ = s or ss. For any $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$,*

$$\limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^*(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!}$$

is finite.

Proof. Take a $\overline{D}_0 \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ such that $\zeta(\overline{D}_0) = \zeta(\overline{D})$ and $\overline{D}_0 \leq \overline{D}$. By Proposition 3.12(1), we have

$$\begin{aligned} & \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^*(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!} \\ & \leq \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^*(m\overline{D}_0; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!} \\ & \quad + (\dim X + 1) \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \left(\frac{3}{2} \|\overline{D} - \overline{D}_0\|_{\ell^1} + \frac{\delta(m\overline{D}_0)}{m} \right) \frac{\dim_{\mathbb{Q}}(mD; m\mathcal{V})}{m^{\dim X} / (\dim X)!} \\ & \leq \widehat{\text{vol}}(\overline{D}_0; \mathcal{V}) + \frac{3}{2} (\dim X + 1) [K : \mathbb{Q}] \|\overline{D} - \overline{D}_0\|_{\ell^1} \text{vol}(D; \mathcal{V}) < +\infty. \end{aligned}$$

\square

Definition 3.4. Given a $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$, we define

$$(3.43) \quad \widehat{\text{vol}}(\overline{D}; \mathcal{V}) := \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!}.$$

By Proposition 3.13, $\widehat{\text{vol}}(\overline{D}; \mathcal{V})$ is finite and

$$(3.44) \quad 0 \leq \widehat{\text{vol}}(\overline{D}; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}_0; \mathcal{V}) \leq \frac{3}{2} (\dim X + 1) [K : \mathbb{Q}] \text{vol}(D; \mathcal{V}) \cdot \|\overline{D} - \overline{D}_0\|_{\ell^1}$$

for every $\overline{D}_0 \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ with $\zeta(\overline{D}_0) = \zeta(\overline{D})$ and $\overline{D}_0 \leq \overline{D}$. Moreover, we can easily observe

$$(3.45) \quad \widehat{\text{vol}}(\overline{D}; \mathcal{V}) = \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^{\text{ss}}(m\overline{D}; m\mathcal{V})}{m^{\dim X + 1} / (\dim X + 1)!}.$$

Proposition 3.14. *Let X be a normal, projective, and geometrically connected K -variety, let $\overline{D} = \left(D, \sum_{v \in M_K} g_v^{\overline{D}}[v] \right) \in \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$, and let $x \in X(\overline{K})$. The infinite sum*

$$\Delta := \sum_{v \in M_K^{\text{fin}}} \sum_{\substack{w \in M_{\kappa(x)}^{\text{fin}} \\ w|v}} [\kappa(x)_w : K_v] g_v^{\overline{D}}(x^w) + \sum_{\sigma: \kappa(x) \rightarrow \mathbb{C}} g_{\infty}^{\overline{D}}(x^{\sigma})$$

then converges, where the limit is taken with respect to the net indexed by all the finite subsets of M_K^{fin} , $x^w \in X_v^{\text{an}}$ is a point corresponding to $(\kappa(x), |\cdot|_w)$, and $x^\sigma \in X_\infty^{\text{an}}$ is a point defined as $\text{Spec}(\mathbb{C}) \xrightarrow{\sigma} \text{Spec}(\kappa(x)) \xrightarrow{x} X$.

Proof. Let $(\mathcal{X}, \overline{\mathcal{D}}) \in \widehat{\text{Mod}}(\overline{D})$ and let $(0, \mathbf{f}) := \overline{D} - \overline{\mathcal{D}}^{\text{ad}}$. We write

$$\mathcal{D} = a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$$

with $a_i \in \mathbb{R}$ and effective Cartier divisors \mathcal{D}_i . Then

$$\begin{aligned} \Delta - \sum_{\sigma: \kappa(x) \rightarrow \mathbb{C}} g_\infty^{\overline{D}}(x^\sigma) \\ &= \sum_{v \in M_K^{\text{fin}}} \sum_{\substack{w \in M_{\kappa(x)}^{\text{fin}}, \\ w|v}} [\kappa(x)_w : K_v] g_v^{(\mathcal{X}, \mathcal{D})}(x^w) + \sum_{v \in M_K^{\text{fin}}} \sum_{\substack{w \in M_{\kappa(x)}^{\text{fin}}, \\ w|v}} [\kappa(x)_w : K_v] f_v(x^w) \\ &= 2 \sum_{i=1}^r a_i \log \#(O_{\kappa(x)}(\mathcal{D}_i)/O_{\kappa(x)}) + \sum_{v \in M_K^{\text{fin}}} \sum_{\substack{w \in M_{\kappa(x)}^{\text{fin}}, \\ w|v}} [\kappa(x)_w : K_v] f_v(x^w) \end{aligned}$$

(see [14, section 2.3]). Let $\varepsilon > 0$. Since $\mathbf{f} \in C_{\ell^1}(X)$, one can find a finite subset $S_0 \subset M_K^{\text{fin}}$ such that

$$\begin{aligned} \left| \sum_{v \in S_1} \sum_{\substack{w \in M_{\kappa(x)}, \\ w|v}} [\kappa(x)_w : K_v] f_v(x^w) - \sum_{v \in S_2} \sum_{\substack{w \in M_{\kappa(x)}, \\ w|v}} [\kappa(x)_w : K_v] f_v(x^w) \right|_\infty \\ \leq [\kappa(x) : K] \sum_{v \in M_K^{\text{fin}} \setminus S_0} \|f_v\|_{\text{sup}} \leq \varepsilon \end{aligned}$$

for every finite subsets S_1, S_2 of M_K^{fin} such that $S_1 \supset S_0$ and $S_2 \supset S_0$. So, by completeness of \mathbb{R} , Δ converges. \square

Definition 3.5. An ℓ^1 -adelic \mathbb{R} -Cartier divisor \overline{D} on X determines a height function $h_{\overline{D}}: X(\overline{K}) \rightarrow \mathbb{R}$ by

$$h_{\overline{D}}(x) := \frac{1}{[\kappa(x) : \mathbb{Q}]} \left(\frac{1}{2} \sum_{v \in M_K^{\text{fin}}} \sum_{\substack{w \in M_{\kappa(x)}^{\text{fin}}, \\ w|v}} [\kappa(x)_w : K_v] g_v^{\overline{D}}(x^w) + \frac{1}{2} \sum_{\sigma: \kappa(x) \rightarrow \mathbb{C}} g_\infty^{\overline{D}}(x^\sigma) \right),$$

which is well-defined by Proposition 3.14 above, and belongs, up to $O(1)$, to the Weil height function corresponding to D . Moreover, from the proof of Proposition 3.14, one deduces

$$(3.46) \quad \sup_{x \in X(\overline{K})} |h_{\overline{D}}(x) - h_{\overline{D}'}(x)| \leq \frac{1}{2} \|\overline{D} - \overline{D}'\|_{\ell^1}$$

for every $\overline{D}, \overline{D}' \in \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ with $\zeta(\overline{D}) = \zeta(\overline{D}')$.

We abbreviate

$$e_{\max}(\overline{D}; \mathcal{V}) := e_{\max}\left(H^0(D; \mathcal{V}), (\|\cdot\|_{v, \text{sup}})_{v \in M_K}\right)$$

(see (3.3)), and define the *essential minimum* of \overline{D} as

$$(3.47) \quad \text{ess.min}_{x \in X(\overline{K})} h_{\overline{D}}(x) = \sup_{Y \subsetneq X} \inf_{x \in (X \setminus Y)(\overline{K})} h_{\overline{D}}(x),$$

where the supremum is taken over all the closed proper subvarieties of X .

Lemma 3.15. *For any $\overline{D} \in \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$, we have*

$$\lim_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{e_{\max}(m\overline{D})}{m} \leq \text{ess.min}_{x \in X(\overline{K})} h_{\overline{D}}(x) < +\infty.$$

Proof. Note that $e_{\max}(\overline{D}) = \min \left\{ \lambda \in \mathbb{R} : \widehat{\Gamma}^s(\overline{D} + (0, 2\lambda[\infty])) \neq \{0\} \right\}$ and

$$\lim_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{e_{\max}(m\overline{D})}{m} = \sup_{m \in \mathbb{Z}_{>0}} \frac{e_{\max}(m\overline{D})}{m}$$

by Fekete's lemma. Let $\lambda \in \mathbb{R}_{\geq 0}$, let $\phi \in \widehat{\Gamma}^s(m\overline{D} + (0, 2\lambda[\infty])) \setminus \{0\}$, and let $Z := \text{Supp}(m\overline{D} + (\phi))$. For every $x \in (X \setminus Z)(\overline{K})$, we have

$$h_{\overline{D}}(x) \geq \frac{1}{2m} \inf_{x \in (X \setminus Z)_{\infty}^{\text{an}}} g_{\infty}^{m\overline{D}}(x) \geq \frac{\lambda}{m}.$$

Hence we have the first inequality.

To show the second inequality, we write

$$\overline{D} = a_1 \overline{D}_1 + \cdots + a_r \overline{D}_r + (0, \mathbf{f})$$

such that $a_i \in \mathbb{R}$, $\overline{D}_i \in \widehat{\text{Div}}(X)$, $\mathbf{f} \in C_{\ell^1}(X)$, and \overline{D}_i are all effective (see [13, Proposition 2.4.2(1)]). We set

$$\overline{D}' := [a_1] \overline{D}_1 + \cdots + [a_r] \overline{D}_r \quad \text{and} \quad \Sigma := \bigcup_{i=1}^r \text{Supp}(\overline{D}_i).$$

By [3, Proposition 2.6], $\{x \in (X \setminus \Sigma)(\overline{K}) : h_{\overline{D}'}(x) \leq C\}$ is Zariski dense in X for a constant C .

If $x \in (X \setminus \Sigma)(\overline{K})$, then $h_{\overline{D}}(x) \leq h_{\overline{D}'}(x) + \|\mathbf{f}\|_{\ell^1}$. Hence

$$\{x \in X(\overline{K}) : h_{\overline{D}}(x) \leq C + \|\mathbf{f}\|_{\ell^1}\} \supset \{x \in (X \setminus \Sigma)(\overline{K}) : h_{\overline{D}'}(x) \leq C\},$$

and the left-hand side is also Zariski dense in X . It implies that the essential minimum is bounded from above by $C + \|\mathbf{f}\|_{\ell^1}$. \square

Lemma 3.16. *For any $(\overline{D}; \mathcal{V}) \in \widehat{\mathbb{D}\text{iv}}_{\mathbb{R}, \mathbb{R}}(X)$, one has*

$$0 \leq \widehat{\text{vol}}(\overline{D}; \mathcal{V}) \leq (\dim X + 1)[K : \mathbb{Q}] \text{vol}(\overline{D}; \mathcal{V}) \max \left\{ \lim_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{e_{\max}(m\overline{D}; m\mathcal{V})}{m}, 0 \right\}.$$

Proof. By Gillet–Soulé's formula [5, Proposition 6], we have

$$0 \leq \widehat{\ell}^s(m\overline{D}; m\mathcal{V}) \leq \max \{e_{\max}(m\overline{D}; m\mathcal{V}), 0\} \cdot \text{rk } H^0(mD; m\mathcal{V}) \\ + 2 \left(\text{rk } H^0(mD) + \log(\text{rk } H^0(mD))! \right)$$

for every $m \in \mathbb{Z}_{>0}$. Therefore,

$$\begin{aligned} 0 \leq \widehat{\text{vol}}(\overline{D}; \mathcal{V}) &\leq (\dim X + 1)[K : \mathbb{Q}] \max \left\{ \lim_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{e_{\max}(m\overline{D}; m\mathcal{V})}{m}, 0 \right\} \\ &\quad \cdot \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\dim_K H^0(mD; m\mathcal{V})}{m^{\dim X} / (\dim X)!} \\ &= (\dim X + 1)[K : \mathbb{Q}] \text{vol}(D; \mathcal{V}) \max \left\{ \lim_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{e_{\max}(m\overline{D}; m\mathcal{V})}{m}, 0 \right\}. \end{aligned}$$

□

Lemma 3.17. *Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$. Let $(\overline{D}_n)_{n \geq 1}$ be an increasing sequence in $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ such that $\zeta(\overline{D}_n) = D$ and such that $\|\overline{D} - \overline{D}_n\|_{\ell^1} \rightarrow 0$ as $n \rightarrow +\infty$. One then has*

$$\widehat{\text{vol}}(\overline{D}; \mathcal{V}) = \lim_{n \rightarrow +\infty} \widehat{\text{vol}}(\overline{D}_n; \mathcal{V}).$$

Proof. Since $(\overline{D}_n)_{n \geq 1}$ is an increasing sequence, we can assume $\overline{D}_n \in \widehat{\text{Div}}_{\mathbb{R}}(X)$ for every $n \geq 1$ by Proposition 3.7. Hence, by (3.44),

$$\left| \widehat{\text{vol}}(\overline{D}; \mathcal{V}) - \widehat{\text{vol}}(\overline{D}_n; \mathcal{V}) \right| \leq \frac{3}{2}(\dim X + 1)[K : \mathbb{Q}] \text{vol}(D; \mathcal{V}) \cdot \|\overline{D} - \overline{D}_n\|_{\ell^1} \rightarrow 0$$

as $n \rightarrow +\infty$. □

Proposition 3.18. *Let $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$. For any $\mathbf{f} \in C_{\ell^1}(X)$, we have*

$$\left| \widehat{\text{vol}}(\overline{D} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\text{vol}}(\overline{D}; \mathcal{V}) \right| \leq \frac{1}{2}(\dim X + 1)[K : \mathbb{Q}] \text{vol}(D; \mathcal{V}) \cdot \|\mathbf{f}\|_{\ell^1}.$$

Proof. Let $(\overline{D}_n)_{n \geq 1}$ be an increasing sequence in $\widehat{\text{Div}}_{\mathbb{R}}(X)$ such that $\zeta(\overline{D}_n) = D$ and such that $\|\overline{D} - \overline{D}_n\|_{\ell^1} \rightarrow 0$ as $n \rightarrow +\infty$, and let $(\mathbf{f}_n)_{n \geq 1}$ be an increasing sequence in $C(X)$ such that $\|\mathbf{f} - \mathbf{f}_n\|_{\ell^1} \rightarrow 0$ as $n \rightarrow +\infty$. By the same arguments as in [14, Proposition 5.1.3], Lemma 3.2 implies

$$\left| \widehat{\text{vol}}(\overline{D}_n + (0, \mathbf{f}_n); \mathcal{V}) - \widehat{\text{vol}}(\overline{D}_n; \mathcal{V}) \right| \leq \frac{1}{2}(\dim X + 1)[K : \mathbb{Q}] \text{vol}(D; \mathcal{V}) \cdot \|\mathbf{f}_n\|_{\ell^1}.$$

By taking $n \rightarrow +\infty$, we have the required assertion by Lemma 3.17. □

3.5. Continuity of the arithmetic volume function. The purpose of this section is to establish the global continuity of the arithmetic volume function over $\widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$ along the directions of ℓ^1 -adelic \mathbb{R} -Cartier divisors (see Theorem 3.21). To begin with, we show the homogeneity of the arithmetic volume function in the following form.

Lemma 3.19. *Let \mathcal{X} be a projective arithmetic variety of dimension $d + 1$ having smooth generic fiber $\mathcal{X}_{\mathbb{Q}}$. Let $\overline{\mathcal{D}} \in \widehat{\text{Div}}_{\mathbb{Q}}(\mathcal{X}; C^{\infty})$ and let $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$ with $\mathcal{V} \geq 0$. For any $p \in \mathbb{Z}_{>0}$, one has*

$$\widehat{\text{vol}}(p\overline{\mathcal{D}}; p\mathcal{V}) = p^{\dim X + 1} \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}).$$

Proof. First, we note the following.

Claim 3.20. *It suffices to show that, to each $\overline{\mathcal{D}} \in \widehat{\text{Div}}_{\mathbb{Q}}(\mathcal{X}; C^\infty)$, one can assign a $q_{\overline{\mathcal{D}}} \in \mathbb{Z}_{>0}$ such that the equality is true for all multiples of $q_{\overline{\mathcal{D}}}$.*

Proof of Claim 3.20. For any $p \in \mathbb{Z}_{>0}$, one has

$$\begin{aligned} \widehat{\text{vol}}(p\overline{\mathcal{D}}; p\mathcal{V}) &= \frac{1}{(q_{\overline{\mathcal{D}}}q_{p\overline{\mathcal{D}}})^{\dim X+1}} \widehat{\text{vol}}\left((pq_{\overline{\mathcal{D}}}q_{p\overline{\mathcal{D}}})\overline{\mathcal{D}}; (pq_{\overline{\mathcal{D}}}q_{p\overline{\mathcal{D}}})\mathcal{V}\right) \\ &= p^{\dim X+1} \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}). \end{aligned}$$

□

By Claim 3.20, it suffices to show the equality for every $p \in \mathbb{Z}_{>0}$ with

$$\overline{\mathcal{D}}' := p\overline{\mathcal{D}} \in \widehat{\text{Div}}(\mathcal{X}; C^\infty).$$

We fix an $\overline{\mathcal{E}} \in \widehat{\text{Div}}(\mathcal{X})$ such that $\overline{\mathcal{E}} \geq 0$ and $\overline{\mathcal{E}} \pm \overline{\mathcal{D}}' \geq 0$. By Theorem 2.6, there is a constant $C > 0$ such that

$$0 \leq \widehat{\ell}^s\left(\mathcal{O}_{\mathcal{X}}(m\overline{\mathcal{D}}' + \overline{\mathcal{E}}); n\mathcal{V}\right) - \widehat{\ell}^s\left(\mathcal{O}_{\mathcal{X}}(m\overline{\mathcal{D}}' - \overline{\mathcal{E}}); n\mathcal{V}\right) \leq Cm^{\dim X}(1 + \log(m))$$

for $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. Hence, for each $r = 1, 2, \dots, p-1$, we obtain

$$\begin{aligned} \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s((pm+r)\overline{\mathcal{D}}'; (pm+r)\mathcal{V})}{(pm+r)^{\dim X+1}/(\dim X+1)!} &\leq \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s\left(\left(m + \frac{r}{p}\right)\overline{\mathcal{D}}'; pm\mathcal{V}\right)}{(pm)^{\dim X+1}/(\dim X+1)!} \\ &= \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s\left(\mathcal{O}_{\mathcal{X}}(m\overline{\mathcal{D}}'); pm\mathcal{V}\right)}{(pm)^{\dim X+1}/(\dim X+1)!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}) &= \max_{0 \leq r < p} \left\{ \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s((pm+r)\overline{\mathcal{D}}'; (pm+r)\mathcal{V})}{(pm+r)^{\dim X+1}/(\dim X+1)!} \right\} \\ &= \limsup_{\substack{m \in \mathbb{Z}, \\ m \rightarrow +\infty}} \frac{\widehat{\ell}^s(pm\overline{\mathcal{D}}'; pm\mathcal{V})}{(pm)^{\dim X+1}/(\dim X+1)!} = \frac{1}{p^{\dim X+1}} \widehat{\text{vol}}(p\overline{\mathcal{D}}; p\mathcal{V}). \end{aligned}$$

□

Theorem 3.21. *Let X be a normal, projective, and geometrically connected K -variety. Let V be a finite-dimensional \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ endowed with a norm $\|\cdot\|_V$, let Σ be a finite set of points on X , and let $B \in \mathbb{R}_{>0}$. For any $\varepsilon \in \mathbb{R}_{>0}$, there exists a $\delta \in \mathbb{R}_{>0}$ such that*

$$\left| \widehat{\text{vol}}(\overline{\mathcal{D}} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\text{vol}}(\overline{\mathcal{E}}; \mathcal{V}) \right| \leq \varepsilon$$

for every $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in V$ with $\max\{\|\overline{\mathcal{D}}\|_V, \|\overline{\mathcal{E}}\|_V\} \leq B$ and $\|\overline{\mathcal{D}} - \overline{\mathcal{E}}\|_V \leq \delta$, $\mathbf{f} \in C_{\ell^1}(X)$ with $\|\mathbf{f}\|_{\ell^1} \leq \delta$, and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ with $\{c_X(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma$.

We need the following.

Proposition 3.22. *Let \mathcal{X} be a projective arithmetic variety of dimension $d + 1$ such that $\mathcal{X}_{\mathbb{Q}}$ is smooth. Let $\overline{V} = (V, \|\cdot\|_V)$ be a couple of a finite-dimensional \mathbb{R} -subspace V of $\widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}; C^\infty)$ and a norm $\|\cdot\|_V$ on V , and let Σ be a finite set of points on \mathcal{X} . There then exists a positive constant $C_{\overline{V}, \Sigma} > 0$ such that*

$$\left| \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}) - \widehat{\text{vol}}(\overline{\mathcal{D}}'; \mathcal{V}) \right| \leq C_{\overline{V}, \Sigma} \max \left\{ \|\overline{\mathcal{D}}\|_V^d, \|\overline{\mathcal{D}}'\|_V^d \right\} \cdot \|\overline{\mathcal{D}} - \overline{\mathcal{D}}'\|_V.$$

for every $\overline{\mathcal{D}}, \overline{\mathcal{D}}' \in V$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(\mathcal{X})$ with $\{c_{\mathcal{X}}(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma$.

Proof. By extending $(V, \|\cdot\|_V)$ if necessary, we may assume that V has a basis $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_r \in \widehat{\text{Div}}(\mathcal{X}; C^\infty)$ such that $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_r$ are all effective. We set

$$\|a_1 \overline{\mathcal{A}}_1 + \dots + a_r \overline{\mathcal{A}}_r\|_1 := |a_1| + \dots + |a_r|$$

for $a_1, \dots, a_r \in \mathbb{R}$, and set

$$\overline{\mathcal{D}} = \mathbf{a} \cdot \overline{\mathcal{A}}, \quad \overline{\mathcal{D}}' = \mathbf{a}' \cdot \overline{\mathcal{A}}, \quad \text{and} \quad \overline{\mathcal{A}} := \overline{\mathcal{A}}_1 + \dots + \overline{\mathcal{A}}_r.$$

If $\mathbf{a}' = 0$, then we can see $\widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}) \leq C \|\mathbf{a}\|_1^{d+1}$ for

$$(3.48) \quad C := \max \left\{ 1, \widehat{\text{vol}}(\overline{\mathcal{A}}) \right\}$$

by using Lemma 3.19, so that we can assume that both \mathbf{a} and \mathbf{a}' are nonzero.

First, we assume $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}^r$ and $\mathbf{b} := \mathbf{a}' - \mathbf{a} \geq 0$. By Theorem 2.6, we get a constant $C' \geq C$ depending only on $\overline{\mathcal{A}}, \Sigma$, and \mathcal{X} such that

$$\begin{aligned} 0 &\leq \widehat{\ell}^s(m\mathcal{O}_{\mathcal{X}}(\overline{\mathcal{D}}_2); m\mathcal{V}) - \widehat{\ell}^s(m\mathcal{O}_{\mathcal{X}}(\overline{\mathcal{D}}_1); m\mathcal{V}) \\ &\leq \widehat{\ell}^s(\mathcal{O}_{\mathcal{X}}(m\mathbf{a} \cdot \overline{\mathcal{A}} + m \max_i \{b_i\} \overline{\mathcal{A}}); m\mathcal{V}) - \widehat{\ell}^s(\mathcal{O}_{\mathcal{X}}(m\mathbf{a} \cdot \overline{\mathcal{A}}); m\mathcal{V}) \\ &\leq C' m^d (\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^d (m\|\mathbf{b}\|_1 + \log(m\|\mathbf{a}\|_1)) \end{aligned}$$

for every $m \in \mathbb{Z}_{>0}$. Hence

$$(3.49) \quad \widehat{\text{vol}}(\overline{\mathcal{D}}_1; \mathcal{V}) \leq \widehat{\text{vol}}(\overline{\mathcal{D}}_2; \mathcal{V}) \leq \widehat{\text{vol}}(\overline{\mathcal{D}}_1; \mathcal{V}) + C' (\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^d \|\mathbf{b}\|_1.$$

For general $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}^r$, we set $\mathbf{a}'' := \max\{\mathbf{a}, \mathbf{a}'\}$ and $\overline{\mathcal{D}}'' := \mathbf{a}'' \cdot \overline{\mathcal{A}}$. By (3.49)

$$\begin{aligned} &\left| \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}) - \widehat{\text{vol}}(\overline{\mathcal{D}}'; \mathcal{V}) \right| \\ &\leq \left| \widehat{\text{vol}}(\overline{\mathcal{D}}''; \mathcal{V}) - \widehat{\text{vol}}(\overline{\mathcal{D}}; \mathcal{V}) \right| + \left| \widehat{\text{vol}}(\overline{\mathcal{D}}''; \mathcal{V}) - \widehat{\text{vol}}(\overline{\mathcal{D}}'; \mathcal{V}) \right| \\ &\leq C' (\|\mathbf{a}\|_1 + \|\mathbf{a}'' - \mathbf{a}\|_1)^d \|\mathbf{a}'' - \mathbf{a}\|_1 + C' (\|\mathbf{a}'\|_1 + \|\mathbf{a}'' - \mathbf{a}'\|_1)^d \|\mathbf{a}'' - \mathbf{a}'\|_1 \\ &\leq 2^d C' \max \left\{ \|\mathbf{a}\|_1^d, \|\mathbf{a}'\|_1^d \right\} \|\mathbf{a} - \mathbf{a}'\|_1. \end{aligned}$$

Therefore, by using Lemma 3.19, we can verify that the estimate is also true for every $\mathbf{a}, \mathbf{a}' \in \mathbb{Q}^r$.

Next, we show the estimate for every $\mathbf{a}, \mathbf{a}' \in \mathbb{R}^r$.

Claim 3.23. *Let $(\mathbf{p}^{(n)})_{n \geq 1}$ be a sequence in \mathbb{Q}^r that converges to $\mathbf{a} \in \mathbb{R}^r$. Then*

$$\lim_{n \rightarrow +\infty} \widehat{\text{vol}}(\mathbf{p}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) = \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}; \mathcal{V}).$$

Proof of Claim 3.23. Let $(\mathbf{b}^{(n)})_{n \geq 1}$ and $(\mathbf{c}^{(n)})_{n \geq 1}$ be two sequences in \mathbb{Q}^r such that

$$b_i^{(1)} \leq b_i^{(2)} \leq \dots \leq b_i^{(n)} \leq \dots \leq a_i \leq \dots \leq c_i^{(n)} \leq \dots \leq c_i^{(2)} \leq c_i^{(1)}$$

and $\lim_{n \rightarrow +\infty} |c_i^{(n)} - b_i^{(n)}| = 0$ for $i = 1, 2, \dots, r$. We then have

$$\begin{aligned} \widehat{\text{vol}}(\mathbf{b}^{(1)} \cdot \overline{\mathcal{A}}; \mathcal{V}) &\leq \widehat{\text{vol}}(\mathbf{b}^{(2)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \leq \dots \leq \widehat{\text{vol}}(\mathbf{b}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \leq \dots \\ &\leq \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}; \mathcal{V}) \\ &\leq \dots \leq \widehat{\text{vol}}(\mathbf{c}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \leq \dots \leq \widehat{\text{vol}}(\mathbf{c}^{(2)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \leq \widehat{\text{vol}}(\mathbf{c}^{(1)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \end{aligned}$$

and

$$\lim_{n \rightarrow +\infty} \left(\widehat{\text{vol}}(\mathbf{c}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) - \widehat{\text{vol}}(\mathbf{b}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \right) = 0$$

by the above arguments. Hence we have the required claim. \square

We choose two sequences $(\mathbf{p}^{(n)})_{n \geq 1}$ and $(\mathbf{q}^{(n)})_{n \geq 1}$ in \mathbb{Q}^r such that $\lim_{n \rightarrow +\infty} \mathbf{p}^{(n)} = \mathbf{a}$ and $\lim_{n \rightarrow +\infty} \mathbf{q}^{(n)} = \mathbf{a}'$, respectively. Then

$$\left| \widehat{\text{vol}}(\mathbf{p}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) - \widehat{\text{vol}}(\mathbf{q}^{(n)} \cdot \overline{\mathcal{A}}; \mathcal{V}) \right| \leq C \max \left\{ \|\mathbf{p}^{(n)}\|_1^d, \|\mathbf{q}^{(n)}\|_1^d \right\} \|\mathbf{p}^{(n)} - \mathbf{q}^{(n)}\|_1$$

for every $n \geq 1$ by the previous argument. Taking $n \rightarrow +\infty$, we obtain the required result. \square

Proof of Theorem 3.21. We may assume that X is smooth. In fact, let $\mu : \tilde{X} \rightarrow X$ be a resolution of singularities of X , and regard V as an \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(\tilde{X})$ via $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X) \rightarrow \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(\tilde{X})$. Since X is normal, we have $\widehat{\text{vol}}(\mu^* \overline{D}; \mathcal{V}) = \widehat{\text{vol}}(\overline{D}; \mathcal{V})$ for every $\overline{D} \in V$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$. Let $\overline{A}_1, \dots, \overline{A}_r \in \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ be a basis for V , put

$$\|a_1 \overline{A}_1 + \dots + a_r \overline{A}_r\|_1 := |a_1| + \dots + |a_r|$$

for $a_1, \dots, a_r \in \mathbb{R}$, and suppose that $\|\cdot\|_V$ is given as $\|\cdot\|_1$. We can easily find a constant $B' \in \mathbb{R}_{>0}$ such that $\text{vol}(D) \leq B'$ for every $\overline{D} \in V$ with $\|\overline{D}\|_1 \leq B$.

We put

$$(3.50) \quad \delta' := \frac{\varepsilon}{2(\dim X + 1)[K : \mathbb{Q}]B'(B + 1)},$$

and fix, for each i , $(\mathcal{X}, \overline{\mathcal{A}}_i) \in \widehat{\text{Mod}}_{\mathbb{R}}(\overline{A}_i)$ such that $\overline{\mathcal{A}}_i \in \widehat{\text{Div}}_{\mathbb{R}}(\mathcal{X}; C^\infty)$ and such that $\|\overline{A}_i - \overline{\mathcal{A}}_i^{\text{ad}}\|_{\ell^1} \leq \delta'$ by using the Stone-Weierstrass theorem and Proposition 3.7. Proposition 3.18 implies that

$$(3.51) \quad \left| \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathbf{A}} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) \right| \leq \frac{1}{2}(\dim X + 1)[K : \mathbb{Q}]B'(\|\mathbf{a}\|_1 + 1)\delta \leq \frac{\varepsilon}{4}$$

holds for every $\mathbf{a} \in \mathbb{R}^r$ with $\|\mathbf{a}\|_1 \leq B$, $\mathbf{f} \in C_{\ell^1}(X)$ with $\|\mathbf{f}\|_{\ell^1} \leq \delta'$, and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$.

Thanks to Proposition 3.22, there is a constant $C_{\overline{\mathcal{A}}, \Sigma} > 0$ such that

$$\left| \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a}' \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) \right| \leq C_{\overline{\mathcal{A}}, \Sigma} \max \{ \|\mathbf{a}\|_1^d, \|\mathbf{a}'\|_1^d \} \|\mathbf{a} - \mathbf{a}'\|_1$$

for every $\mathbf{a}, \mathbf{a}' \in \mathbb{R}^r$ and $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$ with $\{c_X(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma$, so, if we set

$$(3.52) \quad \delta := \min \left\{ \delta', \frac{\varepsilon}{2C_{\Sigma, \Sigma} B^d} \right\},$$

then

$$(3.53) \quad \left| \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a}' \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) \right| \leq \frac{\varepsilon}{2}$$

for every $\mathbf{a}, \mathbf{a}' \in \mathbb{R}^r$ with $\max\{\|\mathbf{a}\|_1, \|\mathbf{a}'\|_1\} \leq B$ and $\|\mathbf{a} - \mathbf{a}'\|_1 \leq \delta$. All in all, we have

$$\begin{aligned} & \left| \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathbf{A}} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a}' \cdot \overline{\mathbf{A}}; \mathcal{V}) \right| \\ & \leq \left| \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathbf{A}} + (0, \mathbf{f}); \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) \right| \\ & \quad + \left| \widehat{\text{vol}}(\mathbf{a} \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a}' \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) \right| + \left| \widehat{\text{vol}}(\mathbf{a}' \cdot \overline{\mathcal{A}}^{\text{ad}}; \mathcal{V}) - \widehat{\text{vol}}(\mathbf{a}' \cdot \overline{\mathbf{A}}; \mathcal{V}) \right| \\ & \leq \varepsilon \end{aligned}$$

as required. \square

Theorem 3.21 implies the following corollaries.

Corollary 3.24. *For a $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$, the following are equivalent.*

- (1) $\widehat{\text{vol}}(\overline{D}; \mathcal{V}) > 0$.
- (2) *For any $\overline{A} \in \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ with $\widehat{\text{vol}}(\overline{A}) > 0$, there exists a $t \in \mathbb{R}_{>0}$ such that $(\overline{D} - t\overline{A}; \mathcal{V}) \geq 0$.*

Corollary 3.25. *For any $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$ and $p \in \mathbb{R}_{>0}$, one has*

$$\widehat{\text{vol}}(p\overline{D}; p\mathcal{V}) = p^{\dim X + 1} \widehat{\text{vol}}(\overline{D}; \mathcal{V}).$$

Proof. We may assume that X is smooth. Let V be a finite-dimensional \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$ such that V has a basis $\overline{A}_1, \dots, \overline{A}_r \in \widehat{\text{Div}}_{\mathbb{Q}}^{\ell^1}(X)$ and such that $\overline{D} = \mathbf{a} \cdot \overline{\mathbf{A}} \in V$ for an $\mathbf{a} \in \mathbb{R}^r$. Let $(\mathbf{b}^{(n)})_{n \geq 1}$ be a sequence in \mathbb{Q}^r that converges to \mathbf{a} . By the Stone–Weierstrass theorem and Proposition 3.7, one finds, for each i , a sequence $((\mathcal{X}_n, \overline{\mathcal{A}}_{in}))_{n \geq 1}$ in $\widehat{\text{Mod}}_{\mathbb{Q}}(\overline{A}_i)$ such that $\overline{\mathcal{A}}_{in} \in \widehat{\text{Div}}_{\mathbb{Q}}(\mathcal{X}_n; C^\infty)$, such that $\overline{\mathcal{A}}_{i1}^{\text{ad}} \leq \overline{\mathcal{A}}_{i2}^{\text{ad}} \leq \dots$, and such that $\|\overline{A}_i - \overline{\mathcal{A}}_{in}^{\text{ad}}\|_{\ell^1} \rightarrow 0$ as $n \rightarrow +\infty$. By Lemma 3.19,

$$\widehat{\text{vol}}(p\mathbf{b}^{(n)} \cdot \overline{\mathcal{A}}_n^{\text{ad}}; p\mathcal{V}) = p^{\dim X + 1} \widehat{\text{vol}}(\mathbf{b}^{(n)} \cdot \overline{\mathcal{A}}_n^{\text{ad}}; \mathcal{V})$$

for $p \in \mathbb{Q}_{>0}$ and $n \geq 1$. Taking $n \rightarrow +\infty$ (Theorem 3.21), we obtain the equality for every $p \in \mathbb{Q}_{>0}$.

To show the corollary, we note that the inequality \leq is obvious. We choose an decreasing sequence $(q_n)_{n \geq 1}$ in $\mathbb{Q}_{>0}$ that converges to p . Then

$$\widehat{\text{vol}}(q_n \overline{D}; p\mathcal{V}) \geq \widehat{\text{vol}}(q_n \overline{D}; q_n \mathcal{V}) = q_n^{\dim X + 1} \widehat{\text{vol}}(\overline{D}; \mathcal{V})$$

for $n \geq 1$. By taking $n \rightarrow +\infty$, we conclude the proof by Theorem 3.21. \square

Corollary 3.26. *For any $(\overline{D}; \mathcal{V}) \in \widehat{\text{Div}}_{\mathbb{R}, \mathbb{R}}^{\ell^1}(X)$ and $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$, one has*

$$\widehat{\text{vol}}(\overline{D} + \widehat{(\phi)}; \mathcal{V}) = \widehat{\text{vol}}(\overline{D}; \mathcal{V}).$$

Proof. We write $\phi = \phi_1^{a_1} \cdots \phi_r^{a_r}$ with $a_i \in \mathbb{R}$ and $\phi_i \in \text{Rat}(X)$. Let V be the \mathbb{R} -subspace of $\widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ generated by ϕ_1, \dots, ϕ_r . For each i , we choose a sequence $(b_i^{(n)})_{n \geq 1}$ in \mathbb{Q} such that $b_i^{(n)} \rightarrow a_i$ as $n \rightarrow +\infty$. By homogeneity (Corollary 3.25), we have

$$\widehat{\text{vol}} \left(\overline{D} + \sum_{i=1}^r b_i^{(n)} \widehat{(\phi_i)}; \mathcal{V} \right) = \widehat{\text{vol}}(\overline{D}; \mathcal{V})$$

for every $n \geq 1$. Taking $n \rightarrow +\infty$, we obtain the required assertion by Theorem 3.21. \square

Corollary 3.27. *For each $\mathcal{V} \in \text{BC}_{\mathbb{R}}(X)$, the arithmetic volume function induces a continuous function $\widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X) \rightarrow \mathbb{R}_{\geq 0}$, $\overline{D} \mapsto \widehat{\text{vol}}(\overline{D}; \mathcal{V})$.*

Proof. By using Corollary 3.26, we can obtain the required map. To show the continuity, let $q : \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X) \rightarrow \widehat{\text{Cl}}_{\mathbb{R}}^{\ell^1}(X)$ be the natural projection and fix a section $\iota' : \text{Cl}_{\mathbb{R}}(X) \rightarrow \widehat{\text{Div}}_{\mathbb{R}}^{\ell^1}(X)$ of ζ . Let V be the image of ι' and let $\|\cdot\|$ be a norm on $\text{Cl}_{\mathbb{R}}(X)$. Set

$$\|\overline{D}\|_{\iota', \|\cdot\|} := \|\zeta(\overline{D})\| + \|\overline{D} - \iota' \circ \zeta(\overline{D})\|_{\ell^1}$$

for $\overline{D} \in V \oplus C_{\ell^1}(X)$, and set $\iota := q \circ \iota'$. We then have $\|\overline{D}\|_{\iota', \|\cdot\|} = \|q(\overline{D})\|_{\iota, \|\cdot\|}$ for every $\overline{D} \in V$. Hence the assertion results from Theorem 3.21. \square

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